Photoelectron counting in quantum optics

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Manchester

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Overview: photons, photon-counting, fluctuations

Counting photons

- Counting photons, but...
- ...'the eternal question: what is a photon'.
- 'What is light ?'

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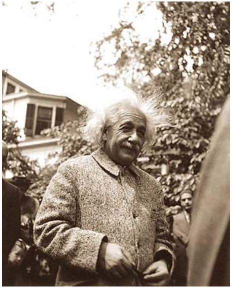

Einstein 1951: '...these days every fool pretends to know what a *photon* is. I have been thinking about this for the whole of my life, and I haven't found the answer'.

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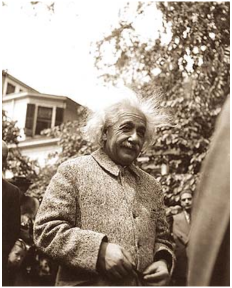

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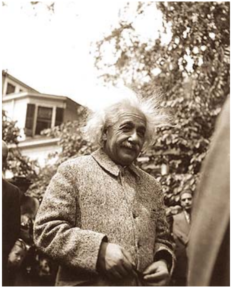

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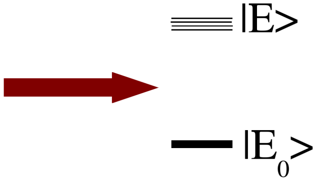
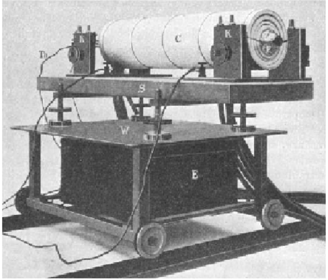
...photon as gauge-boson of QED .

Overview: photons, photon-counting, fluctuations

Irony of history: quantum mechanics



- ‘No photons’ for the photoelectric effect.
- Quantum mechanics was discovered in its own classical limit.



Overview: photons, photon-counting, fluctuations

Irony of history: quantum optics

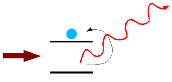
- Big breakthrough: Hanbury Brown, Twiss experiment: intensity correlations, 'photon bunching'.
- Correlation functions ($a^\dagger$ creates cavity mode):

$$G^{(1)}(t, t+\tau) = \langle a^\dagger(t) a(t+\tau) \rangle \quad (1)$$

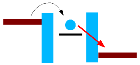
$$G^{(2)}(t, t+\tau) = \langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t) \rangle. \quad (2)$$

- But not yet a complete triumph for quantum optics...

Triumph came with *resonance fluorescence*: photon antibunching,



Resonance fluorescence



CB dot, tunneling

Overview: photons, photon-counting, fluctuations

Photon counting: some issues

- Count photo-electrons instead of photons.
- Counting statistics: correct theory for

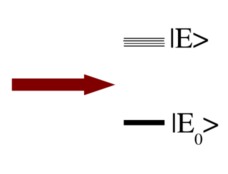
$$p_n(t, t+T) \quad \text{probability for } n \text{ photo-electrons in } [t, t+T).$$

- Detector back-action. System-bath problem 'with two baths'.
- ... no entirely trivial!

Semiclassical theory for $p_n(t, t+T)$: Mandel formula

Photodetector model: ionize single atom

- Classical electromagnetic field, vector potential $\mathbf{A}(\mathbf{r})e^{-i\omega t} + \mathbf{A}^*(\mathbf{r})e^{i\omega t}$.



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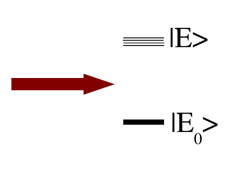

Probability $p_1(t, t+\Delta t)$ of one count: Fermi's Golden Rule

$$\begin{aligned} p_1(t, t+\Delta t) &= \int_0^\infty dE \nu(E) \left| \langle E | \frac{e}{m} \mathbf{p}\mathbf{A}(\mathbf{r}) | E_0 \rangle \right|^2 D_{\Delta t}(E - E_0 - \omega) \\ &= \eta I(\mathbf{r}) \Delta t, \quad I(\mathbf{r}) = |A(\mathbf{r})|^2 (\text{intensity}). \end{aligned} \tag{1}$$

- $D_{\Delta t}(\varepsilon) \equiv \left([\sin \frac{1}{2}\varepsilon \Delta t] / [\frac{1}{2}\varepsilon] \right)^2$, $\Delta t \to 0$. Polarisation $\mathbf{A}(\mathbf{r}) = \vec{\varepsilon} A(\mathbf{r})$.

Mandel formula: many counts

How to obtain probability of n transitions $p_n(t, t+T)$

- Short-time probability $p_1(t, t+\Delta t) = \eta I(\mathbf{r})\Delta t$ for *single* electron transition ($\eta I(\mathbf{r})$ transition rate).
- Long-time probability of n transitions $p_n(t, t+T) \leftrightarrow n$ electrons.

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- Individual transitions are statistically independent...
- $\rightsquigarrow$ Poisson distribution.
- Characterized by average $\bar{n}$ only $\rightsquigarrow$

$$p_n(t, t+T) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \quad \bar{n} = \eta I(\mathbf{r}) T. \tag{2}$$

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- Markovian master equation for probabilities. $p_n(t) \equiv p_n(0, t)$,

$$p_n(t+dt) = p_n(t) \times [1 - \eta I(\mathbf{r})dt] + p_{n-1}(t) \times \eta I(\mathbf{r})dt \quad (2)$$

$$\frac{d}{dt} p_n(t) = \eta I(\mathbf{r})[p_{n-1}(t) - p_n(t)]. \quad (3)$$

- Generating function $G(s,t) \equiv \sum_{n=0}^{\infty} s^n p_n(t)$,
  $\partial_t G(s,t) = \eta I(\mathbf{r})(s-1)G(s,t)$.
- Solve with $p_0(0) = 1$, $p_n(0) = 0$, $n > 0$, $G(s,0) = 1$.
- Thus $G(s,t) = \exp[\eta I(\mathbf{r})t(s-1)] = \sum_{n=0}^{\infty} s^n \frac{\bar{n}^n}{n!} e^{-\bar{n}}$, $\bar{n} = \eta I(\mathbf{r})t$.

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SUMMARY so far:

- Classical photo-electron counting formula (Mandel formula)

$$p_n(t, t+T) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \quad \bar{n} = \eta I(\mathbf{r}) T.$$

- Poisson process.
- Generating function $G(s,t) \equiv \sum_{n=0}^{\infty} s^n p_n(t) = \exp[\eta I(\mathbf{r}) t(s-1)]$.
- Nothing said here about PHOTONS! This is a DETECTOR theory.

'Quantum Mandel formulas'

Kelley-Kleiner, Carmichael, etc. version

- $p_n(t, t+T) = \langle : \frac{\hat{\Omega}^n}{n!} e^{-\hat{\Omega}} : \rangle$ with $\hat{\Omega} \equiv \xi \int_t^{t+T} dt' \hat{E}^-(t')\hat{E}^+(t')$.
- No backaction of detector on field.
- 'Non-absorbed photons escape, open system.'
- Typically many field degrees of freedom, field is a 'BIG QUANTUM SYSTEM'.

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Mollow; Scully/Lamb; Srivinas/Davies; Ueda etc. version

- Backaction of detector leads to damping (continuous measurement) of the field.
- 'Eventually all photons absorbed, closed system.'
- Typically few field degrees of freedom, field is a 'SMALL QUANTUM SYSTEM'

Scully-Lamb photodetector

M. Scully, W. Lamb Jr., Phys. Rev. **179**, 368 (1969)

- ‘Photon statistics’ means (reduced) density operator $\rho(t)$ of a light field (more generally: boson field).
- ‘Photon statistics’ is inferred by photoelectric counting techniques.

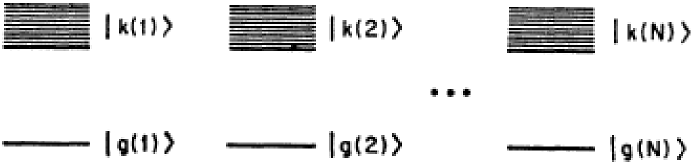
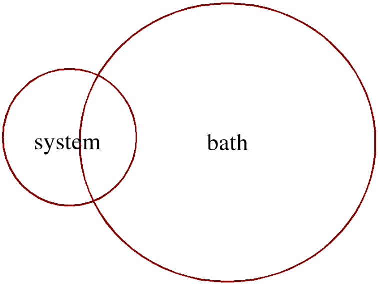


FIG. 1. Pictorial representation of photodetector consisting of N-independent atoms. Each atom in detector has a ground state $|g\rangle$ and continuum of excited states $|k\rangle$. Atoms are labeled by indexing atomic state with particle number, e.g., $|k(m)\rangle$.

System-bath theory

Divide ‘total universe’ into system S and bath B,

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_{\rm S} + \mathcal{H}_{\rm B} + \mathcal{H}_{\rm SB} \\ &\equiv \mathcal{H}_0 + V, \quad V \equiv \mathcal{H}_{\rm SB}. \end{aligned} \quad (2)$$



Total density matrix $\chi(t)$ obeys the Liouville-von-Neumann equation

$$\frac{d}{dt}\chi(t) = -i[\mathcal{H}, \chi(t)]. \quad (3)$$

Master equation

- Effective density matrix of the system $\rho(t) \equiv \mathrm{Tr}_B[\chi(t)]$.
- Interaction picture with respect to H_0,

$$\frac{d}{dt}\tilde{\rho}(t) = -i\mathrm{Tr}_B[\tilde{V}(t), \chi(t=0)] - \int_0^t dt' \mathrm{Tr}_B[\tilde{V}(t), [\tilde{V}(t'), \tilde{\chi}(t')]].$$

- Born approximation, $\tilde{\chi}(t') \approx R_0 \otimes \tilde{\rho}(t')$, R_0 bath density matrix.
- System-bath interaction as $V = \sum_k S_k \otimes B_k$,
- Bath correlation functions $C_{kl}(t, t') \equiv \mathrm{Tr}_B\left[\tilde{B}_k(t)\tilde{B}_l(t')R_0\right]$, $\mathrm{Tr}_B \tilde{B}_k(t)R_0 = 0$.

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$$\begin{aligned} \frac{d}{dt}\tilde{\rho}(t) &= -\int_0^t dt' \sum_{kl} \Big[C_{kl}(t-t') \left\{ \tilde{S}_k(t)\tilde{S}_l(t')\tilde{\rho}(t') - \tilde{S}_l(t')\tilde{\rho}(t')\tilde{S}_k(t) \right\} \\ &+ C_{lk}(t'-t) \left\{ \tilde{\rho}(t')\tilde{S}_l(t')\tilde{S}_k(t) - \tilde{S}_k(t)\tilde{\rho}(t')\tilde{S}_l(t') \right\} \Big]. \end{aligned} \quad (4)$$

Scully-Lamb Photodetector

Detector model

- *System*: single photon mode a and N detector single level 'quantum dots' j with one ($|1\rangle_j$) or zero ($|0\rangle_j$) electrons.
- Photon absorption empties dots into *bath*: leads j, $c^\dagger_{\alpha j}|vac\rangle$.

$$\mathcal{H}_{\mathrm{SB}} = \sum_{\alpha j} \left(V^j_\alpha c^\dagger_{\alpha j} |0\rangle_j \langle 1| a + \bar{V}^j_\alpha c_{\alpha j} |1\rangle_j \langle 0| a^\dagger \right) \equiv \sum_k S_k \otimes B_k . \tag{5}$$

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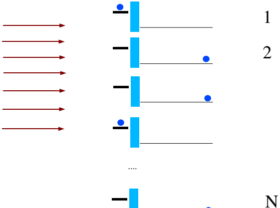

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Master equation: trace out the leads

- Terms $C_{kl}(t-t')\tilde{S}_k(t)\tilde{S}_l(t')\tilde{\rho}(t')$; $C_{kl}(t-t') = \langle \tilde{B}_k(t)\tilde{B}_l(t')\rangle$.
- 'Broadband detection' at all energies, $\sum_\alpha |V^j_\alpha|^2\delta(\varepsilon - \varepsilon_{\alpha j}) = \nu$.

$$\frac{d}{dt}\tilde{\rho}_t = -\pi\nu\sum_j\left\{|1\rangle_j\langle 1|a^\dagger a\tilde{\rho}_t + \tilde{\rho}_t a^\dagger a|1\rangle_j\langle 1| - 2|0\rangle_j\langle 1|a\tilde{\rho}_t a^\dagger|1\rangle_j\langle 0|\right\}.$$

Scully-Lamb Photodetector

State with m excitations

- Detector states $|m;\lambda\rangle \equiv \hat{\Pi}_\lambda |0\rangle_1 ... |0\rangle_m |1\rangle_{m+1} ... |1\rangle_N$. Permutations
- m-resolved field 'pseudo' density matrix $\tilde{\rho}_t^{(m)} \equiv \sum_\lambda \langle m;\lambda|\tilde{\rho}_t|m;\lambda\rangle$.

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$$\sum_j \langle m;\lambda|\tilde{\rho}_t|1\rangle_j\langle 1|m;\lambda\rangle = (N-m)\langle m;\lambda|\tilde{\rho}_t|m;\lambda\rangle$$

$$\sum_j \langle m;\lambda|0\rangle_j\langle 1|\tilde{\rho}_t|1\rangle_j\langle 0|m;\lambda\rangle = \sum_{\lambda'}^{m\text{terms}} \langle m-1;\lambda'|\tilde{\rho}_t|m-1;\lambda'\rangle$$

$$\frac{d}{dt}\tilde{\rho}_t^{(m)} = -\pi\nu \left\{ (N-m)\left[a^\dagger a\tilde{\rho}_t^{(m)} + \tilde{\rho}_t^{(m)} a^\dagger a\right] - 2(N-m+1)a\tilde{\rho}_t^{(m-1)}a^\dagger \right\}$$

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$N \gg m$, $\gamma \equiv 2\pi N \nu \rightsquigarrow$

$$\frac{d}{dt}\rho_t^{(m)} = -i[\mathcal{H}_{\mathrm{F}}, \rho_t^{(m)}] - \frac{\gamma}{2}\left(a^\dagger a \rho_t^{(m)} + \rho_t^{(m)} a^\dagger a - 2a\rho_t^{(m-1)}a^\dagger\right). \quad (5)$$

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- Now counting statistics as $p_m(t) \equiv \mathrm{Tr}\rho_t^{(m)}$!

Jump super-operator J, $J\rho \equiv \gamma a \rho a^\dagger$, time evolution generator $\mathcal{L}_0$

- Define $\mathcal{L}_0\rho \equiv Y\rho + \rho Y^\dagger$ with $Y \equiv -i\mathcal{H}_{\mathrm{F}} - \frac{\gamma}{2} a^\dagger a$.

$$\boxed{\dot{\rho}_t^{(m)} = \mathcal{L}_0 \rho_t^{(m)} + J\rho_t^{(m-1)}.} \quad (6)$$

Summary: counting statistics in Scully-Lamb detector model

m-resolved field density matrix

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Generating operator $\hat{G}(s,t)$

- Define $\hat{G}(s,t) \equiv \sum_{m=0}^{\infty} s^m \rho_t^{(m)}$, s: *counting variable*.
- Usually s complex, e.g. $s = e^{i\phi}$ with real ϕ.
- Infinite set of master equations now becomes a single equation,

$$\boxed{\frac{\partial}{\partial t} \hat{G}(s,t) = (\mathcal{L}_0 + sJ) \hat{G}(s,t).} \tag{7}$$

Solve $\frac{d}{dt}\hat{G} = -i[\mathcal{H}_{\mathrm{F}}, \hat{G}] - \frac{\gamma}{2}\left(a^\dagger a\hat{G} + \hat{G}a^\dagger a - 2sa\hat{G}a^\dagger\right)$

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P-representation in harmonic oscillator Hilbert space

- Glauber introduced coherent states $|z\rangle$, $a|z\rangle = z|z\rangle$.
- Glauber-Sudarshan representation of operators such as $\hat{G}$ as $\hat{G} = \int d^2zP(\hat{G}; z, z^*)|z\rangle\langle z|$.
- z and z^* independent variables. Short form $P(z)$ instead $P(\hat{G}; z, z^*)$.
- Rules $a\hat{G}a^\dagger \leftrightarrow zz^*P(z)$, $a^\dagger a\hat{G} \leftrightarrow (z^* - \partial_z)P(z)$, $\hat{G}a^\dagger a \leftrightarrow (z - \partial_{z^*})P(z)$.

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PDE for P-function of generating operator

- Field Hamiltonian $\mathcal{H}_{\mathrm{F}} = \Omega a^\dagger a$.

$$
\begin{aligned}
\frac{\partial}{\partial t} P_s(z,t) &= \left[-yz\partial_z - y^* z^* \partial_{z^*} + \gamma(1 + |z|^2(s-1))\right] P_s(z,t) \\
y &\equiv -i\Omega - \frac{\gamma}{2}. \qquad (8)
\end{aligned}
$$

Solve $\frac{\partial}{\partial t} P_s = \left[-yz\partial_z - y^* z^* \partial_{z^*} + \gamma(1 + |z|^2(s-1)) \right] P_s$

Case $s = 1$: simply damped harmonic oscillator

- 1st order PDE's are solved by method of characteristics

$$P_1(z,t) = e^{\gamma t} P^{(0)} \left(z e^{i(\Omega - i\gamma/2)t} \right) \tag{9}$$

Example ($G(s, t=0) \equiv \rho^{(0)}(t=0) = |z_0\rangle\langle z_0|$)

$$\begin{aligned} P_1(z, t=0) &= \delta^{(2)}(z - z_0) \rightsquigarrow \qquad (10) \\ P_1(z, t>0) &= e^{\gamma t} \delta^{(2)} \left(z e^{i(\Omega - i\gamma/2)t} - z_0 \right) = \delta^{(2)} \left(z - z_0 e^{-i(\Omega - i\gamma/2)t} \right) \end{aligned}$$

(two-dimensional Delta-function!). State spirals into the origin.

Solve $\frac{\partial}{\partial t} P_s = \left[-yz\partial_z - y^* z^* \partial_{z^*} + \gamma(1 + |z|^2(s-1)) \right] P_s$

Arbitrary s:
$P_s(z,t) = e^{\gamma t} P^{(0)}\left(z e^{i(\Omega - i\gamma/2)t}\right) \exp\{-|z|^2(s-1)(1-e^{\gamma t})\}$

- Now $\mathrm{Tr}\hat{G}(s,t) \equiv \sum_{m=0}^{\infty} s^m \mathrm{Tr}\rho_t^{(m)}$, read off *photoelectron counting distribution* $p_m(t) \equiv \mathrm{Tr}\rho_t^{(m)}$.

$$\begin{aligned}
\mathrm{Tr}\hat{G}(s,t) &= \int d^2 z P_s(z,t) = \int d^2 z P^{(0)}(z) e^{-|z|^2(s-1)(e^{-\gamma t}-1)} \\
&= \sum_{m=0}^{\infty} s^m \int d^2 z P^{(0)}(z) \frac{\left(|z|^2 \eta_t\right)^m}{m!} e^{-|z|^2 \eta_t}, \quad \eta_t \equiv 1 - e^{-\gamma t}.
\end{aligned}$$

- Use normal ordering property of P-representation,

$$\boxed{p_m(t) = \mathrm{Tr}\rho(0) : \frac{\left(a^\dagger a \eta_t\right)^m}{m!} e^{-a^\dagger a \eta_t} :, \quad \eta_t \equiv 1 - e^{-\gamma t}} \qquad (11)$$

Single-mode counting formula: discussion of $p_m(t) = \mathrm{Tr}\rho(0) : \frac{\left(a^\dagger a\eta_t\right)^m}{m!} e^{-a^\dagger a\eta_t} :, \quad \eta_t \equiv 1 - e^{-\gamma t}$

- Coherent state $\rho(0) = |z_0\rangle\langle z_0| \rightsquigarrow$

$$p_m(t) = \frac{(\langle n\rangle\eta_t)^m}{m!} e^{-\langle n\rangle\eta_t}.$$

  - Poisson-distribution.
  - Average $\langle n\rangle \equiv \langle a^\dagger a\rangle = |z_0|^2$.
  - Coincides with semiclassical Mandel formula for $\gamma t \ll 1$.

- Fock-state $\rho(0) = |n\rangle\langle n| \rightsquigarrow$

$$p_m(t) = \binom{n}{m} \eta_t^m (1-\eta_t)^{n-m}, \quad n \geq m.$$

  - Bernoulli-distribution.
  - m successful events (counts), $n-m$ failures (no counts) regardless of order.

Summary part 1

Done so far

- Photon counting: photo-electron counting.
- Semiclassical Mandel formula.
- Photo-detector theory: Scully/Lamb.
- Some techniques: quantum master equations, $P-$representation, counting variables and generating functions/operators.

Summary part 1

Done so far

- Photon counting: photo-electron counting.
- Semiclassical Mandel formula.
- Photo-detector theory: Scully/Lamb.
- Some techniques: quantum master equations, $P-$representation, counting variables and generating functions/operators.

Still to do

- More general situations.
- Sources, fields, and detectors.

Photoelectron counting in quantum optics

Tobias Brandes

Manchester

7th January 2006

Revision: towards a counting formula in quantum optics

- Mandel (Poissonian)

$$p_n(t, t+T) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \quad \bar{n} = \eta I(\mathbf{r}) T.$$

  - Classical field with intensity $I(\mathbf{r})$. Golden rule (photo-electric effect).

- Mollow, Scully-Lamb single mode

$$p_n(0, t) = \mathrm{Tr}\rho(0) : \frac{1}{n!} \left(a^\dagger a \eta_t\right)^n \exp(-a^\dagger a \eta_t) :, \quad \eta_t \equiv 1 - e^{-\gamma t}.$$

  - Correctly describes detector backaction. 'Closed system'. Free cavity fields only, no sources.

- 'Quantum Mandel', Kelley-Kleiner

$$p_n(t, t+T) = \langle : \frac{\hat{\Omega}^n}{n!} e^{-\hat{\Omega}} : \rangle.$$

  - Heisenberg operators, $\Omega \equiv \xi \int_t^{t+T} dt' \hat{E}^-(t') \hat{E}^+(t')$.
  - Not correct for long times. 'Open system'. Various generalisations on the market.

Coherence functions
Definitions

Notation $x = (\mathbf{r}, t)$.

$$G^{(1)}(x, x') \equiv \langle E^{(-)}(x) E^{(+)}(x') \rangle \tag{12}$$

$$G^{(2)}(x_1, x_2, x_2', x_1') \equiv \langle E^{(-)}(x_1) E^{(-)}(x_2) E^{(+)}(x_2') E^{(+)}(x_1') \rangle. \tag{13}$$

- Based on photon *absorption* $\rightsquigarrow$ intensity $\langle I(x) \rangle = G^{(1)}(x, x)$.
- $G^{(1)}$ describes first order coherence: Mach-Zehnder (Young, Michelson) interference.
- $G^{(2)}$ describes second order coherence: Hanbury Brown, Twiss.

Coherence functions

Special cases, normalised versions, single-mode example $H = \omega a^\dagger a$

$$G^{(1)}(t, t+\tau) \equiv \langle E^{(-)}(t) E^{(+)}(t+\tau)\rangle \quad (12)$$

$$G^{(2)}(t, t+\tau) \equiv \langle E^{(-)}(t) E^{(-)}(t+\tau) E^{(+)}(t+\tau) E^{(+)}(t)\rangle \quad (13)$$

$$g^{(2)}(t, t+\tau) \equiv \frac{G^{(2)}(t, t+\tau)}{G^{(1)}(t,t) G^{(1)}(t+\tau, t+\tau)} \quad (14)$$

$$\text{number state} \quad \rho(0) = |n\rangle\langle n| \rightsquigarrow g^{(2)}(\tau) = \frac{n(n-1)}{n^2} = 1 - \frac{1}{n}$$

$$\text{coherent state} \quad \rho(0) = |z\rangle\langle z| \rightsquigarrow g^{(2)}(\tau) = \frac{z^* z^* z z}{|z^* z|^2} = 1. \quad (15)$$

Definition (bunching, antibunching; sub/super-Poissonian)

- Bunching: $g^{(2)}(\tau) < g^{(2)}(0)$, anti-bunching $g^{(2)}(\tau) > g^{(2)}(0)$.
- Super-P. $g^{(2)}(0) > 1$, sub-P. $g^{(2)}(0) < 1$: relation to $p_n(t, t+T)$.

Coherence functions

Example for bunching: cavity mode $a^\dagger$ in thermal bath (temperature β^{-1})

- Mode angular frequency ω, damping κ.
- Master equation.
- Use *quantum regression theorem*.
- Long-time limit, $t \to \infty$, $n_{\mathrm{B}} = [e^{\beta\omega} - 1]^{-1}$

$$\lim_{t\to\infty} \langle a^\dagger(t) a(t+\tau)\rangle = n_{\mathrm{B}} e^{-(\kappa+i\omega)\tau} \quad (12)$$

$$\lim_{t\to\infty} \langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t)\rangle = n_{\mathrm{B}}^2 \left(1 + e^{-2\kappa\tau}\right). \quad (13)$$

- Thus, $g^{(2)}(\tau) = 1 + e^{-2\kappa\tau}$ and $g^{(2)}(\tau) < g^{(2)}(0)$: photon bunching.

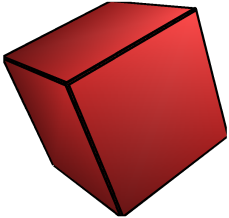
(cf. Carmichael book etc.)

Now from single mode $a^\dagger$ to many modes $a_Q^\dagger$.

Quantization of Maxwell's equations

- Vector potential in Coulomb gauge.
- Fourier expansion into field modes $\mathbf{u}_Q(\mathbf{r})$, mode index Q.

$$(\nabla^2 + \omega_Q^2)\mathbf{u}_Q(\mathbf{r}) = 0.$$



- Quantization, annihilation operator a_Q, creation operator $a_Q^\dagger$.
- Electric field operator

$$\mathbf{E}(\mathbf{r}) = i\sum_Q \left(\frac{\hbar\omega_Q}{2\varepsilon_0}\right)^{1/2} \mathbf{u}_Q(\mathbf{r})a_Q + H.c. = \mathbf{E}^{(+)}(\mathbf{r}) + \mathbf{E}^{(-)}(\mathbf{r}).$$

The most basic case: two-level atom...

Spontaneous emission from a two-level atom

Two-level atom with states $|1\rangle$, $|0\rangle$

$$H = \frac{\omega_0}{2}\sigma_z + \sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q \omega_Q a_Q^\dagger a_Q. \qquad (14)$$

Pauli matrices, photon creation operators $a_Q^\dagger$.

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Pauli matrices, photon creation operators $a_Q^\dagger$.

Algebra of Pauli matrices

$$\begin{aligned}
\sigma_x &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\sigma_- &\equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\sigma_\pm &= \frac{1}{2}(\sigma_x \pm i\sigma_y), \quad \sigma_x = \sigma_+ + \sigma_-, \quad \sigma_y = -i(\sigma_+ - \sigma_-) \\
[\sigma_+, \sigma_-] &= \sigma_z, \quad [\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm. \qquad (15)
\end{aligned}$$

Spontaneous emission from a two-level atom

Two-level atom with states $|1\rangle$, $|0\rangle$

$$H = \frac{\omega_0}{2}\sigma_z + \sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q \omega_Q a_Q^\dagger a_Q. \qquad (14)$$

Pauli matrices, photon creation operators $a_Q^\dagger$.

- Schrödinger equation for total wave function

$$|\Psi(t)\rangle = c(t)|1\rangle|\mathrm{vac}\rangle + \sum_Q b_Q(t)|0\rangle a_Q^\dagger|\mathrm{vac}\rangle, \quad c(0) = 1 \qquad (15)$$

- Can be solved (Wigner-Weisskopf) within some approximations. In particular, $c(t) = e^{-\Gamma t/2 - i\omega_0 t}$.
- No re-absorption of any emitted photon $\leftrightarrow$ single mode model (only one Q, Jaynes-Cummings Hamiltonian, revivals).

Spontaneous emission from a two-level atom

Two-level atom with states $|1\rangle$, $|0\rangle$

$$H = \frac{\omega_0}{2}\sigma_z + \sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q \omega_Q a_Q^\dagger a_Q. \quad (14)$$

Pauli matrices, photon creation operators $a_Q^\dagger$.

- Electric field $\mathbf{E}^{(+)}(\mathbf{r},t) = \mathbf{E}_f^{(+)}(\mathbf{r},t) + \mathbf{E}_s^{(+)}(\mathbf{r},t)$, source field in terms of *source* operators
- Heisenberg EOM $\dot{a}_Q(t) = -i\omega_Q a_Q(t) - i\gamma_k \sigma_-(t) \rightsquigarrow$

$$a_Q(t) = a_Q e^{-i\omega_Q t} - i\gamma_Q \int_0^t dt' \sigma_-(t') e^{-i\omega_Q(t-t')}. \quad (15)$$

Spontaneous emission from a two-level atom

Two-level atom with states $|1\rangle$, $|0\rangle$

$$H = \frac{\omega_0}{2}\sigma_z + \sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q \omega_Q a_Q^\dagger a_Q. \tag{14}$$

Pauli matrices, photon creation operators $a_Q^\dagger$.

- Field at the detector in terms of atom dipole operator

$$\underline{\mathbf{E}_s^{(+)}(\mathbf{r}, t)} = \int_0^t dt' \left[\sum_Q \mathbf{f}_Q(\mathbf{r}) e^{-i\omega_Q(t-t')}\right] \sigma_-(t') \tag{15}$$

$$\approx \int_0^t dt' \left[\mathcal{E}(\hat{\mathbf{r}})\delta(t - t' - r/c)\right] \sigma_-(t') = \underline{\mathcal{E}(\hat{\mathbf{r}})\sigma_-(t - r/c)}.$$

- Note dipole form of $\mathcal{E}(\hat{\mathbf{r}})$.

Spontaneous emission from a two-level atom

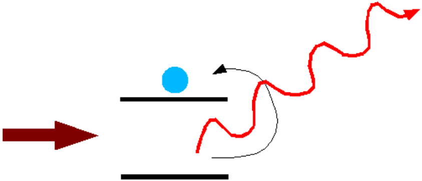
Two-level atom with states $|1\rangle$, $|0\rangle$

$$H = \frac{\omega_0}{2}\sigma_z + \sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q \omega_Q a_Q^\dagger a_Q. \qquad (14)$$

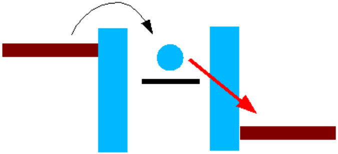
Pauli matrices, photon creation operators $a_Q^\dagger$.

- Not too much can be learned here: transient process, exponentially decaying probability.
- We want to describe stationary processes $\rightsquigarrow$ 'driven spontaneous emission' (resonance fluorescence).
- Analogy to tunneling of a single electron from a single level quantum dot.

Resonance fluorescence: analogy to single electron tunneling



Resonance fluorescence



CB dot, tunneling

Resonance fluorescence model

- Spontaneous emission from TLS plus driving with classical field $E\cos(\omega_L t)$, *Rabi-frequency* $\Omega \equiv dE/\hbar$, d dipole moment.

$$\mathcal{H}_t \equiv \mathcal{H}_{\rm SE} + \frac{\Omega}{2}\left(e^{-i\omega_L t}\sigma_+ + e^{i\omega t}\sigma_-\right), \quad \text{(RWA)}. \tag{15}$$

- Time-dependent unitary trafo leaves Liouville-v.Neumann equation invariant

$$\bar{\mathcal{H}}_t \equiv -iU_t^\dagger \frac{\partial U_t}{\partial t} + U_t^\dagger \mathcal{H}_t U_t, \quad \bar{\rho}_t \equiv U_t^\dagger \rho_t U_t. \tag{16}$$

- The form $U_t = \exp(-i\hat{N}_{\rm F}\omega_L t){\rm diag}(e^{-i\omega_L t}, 1)$ leads to $(\omega_0 = \omega_L)$

$$\boxed{\bar{\mathcal{H}}_t \equiv \frac{\Omega}{2}(\sigma_+ + \sigma_-) + \sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q (\omega_Q - \omega_L) a_Q^\dagger a_Q} \tag{17}$$

Master equation for TLS-'source' density operator ρ_t

$$\dot{\rho}_t = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, \rho_t] - \beta\left(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+\right)$$

- Spontaneous emission rate $\beta = \pi \sum_Q \gamma_Q^2 \delta(\omega_L - \omega_Q)$, effect of driving in β neglected ($\leftrightarrow$ 'intra-collisional field effect).
- Compare with our previous detector equation, $\dot{\rho}_t^{(m)} = -i[\mathcal{H}_{\mathrm{F}}, \rho_t^{(m)}] - \frac{\gamma}{2}\left(a^\dagger a \rho_t^{(m)} + \rho_t^{(m)} a^\dagger a - 2a\rho_t^{(m-1)} a^\dagger\right)$.

Master equation for TLS-'source' density operator ρ_t

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- Remember spontaneous emission: field at the detector in terms of atom dipole operator,

$$\mathbf{E}_s^{(+)}(\mathbf{r}, t) \approx \mathcal{E}(\hat{\mathbf{r}})\sigma_-(t - r/c).$$

- Thus $a \sim \mathbf{E}_s^{(+)} \sim \sigma_-$.
- $\rightsquigarrow$ detector photon absorption $\sim$ electron jumps from up to down, σ_-.

Cook's 'counting at the source'

R. J. Cook PRA **23**, 1243 (1981)

n-resolved master equation for resonance fluorescence of driven TLS

$$\dot{\rho}_t^{(n)} = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, \rho_t^{(n)}] - \beta\left(\sigma_+\sigma_-\rho_t^{(n)} + \rho_t^{(n)}\sigma_+\sigma_- - 2\sigma_-\rho_t^{(n-1)}\sigma_+\right)$$

- Splitting up $\rho_t = \sum_{n=0}^{\infty} \rho_t^{(n)}$, n photon emissions.

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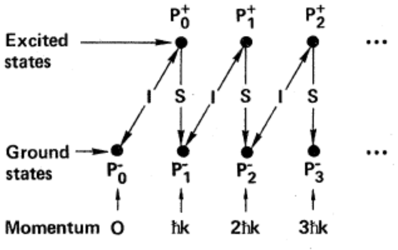

- Cook's original idea: momentum transfers between atom and driving field.
- Count number of discrete displacements $n\hbar k$.
- Alternatively, count number of spontaneous emission events.

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- Splitting up $\rho_t = \sum_{n=0}^{\infty} \rho_t^{(n)}$, n photon emissions.

- Jump super-operator J with $J\rho = 2\beta\sigma_-\rho\sigma_+ = 2\beta|-\rangle\langle+|\rho|+\rangle\langle-|$.
- Generating operator as usual, $G(s,t) \equiv \sum_{n=0}^{\infty} s^n\rho_t^{(n)}$; *counting variable* s.
- Counting statistics as $p_n(0,t) = \mathrm{Tr}\rho_t^{(n)}$.
- Photons are integrated out: just 4 by 4 equation

$$\partial_t G = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, G] - \beta\left(\sigma_+\sigma_- G + G\sigma_+\sigma_- - 2s\sigma_- G\sigma_+\right).$$

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$$\dot{\rho}_t^{(n)} = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, \rho_t^{(n)}] - \beta\left(\sigma_+\sigma_-\rho_t^{(n)} + \rho_t^{(n)}\sigma_+\sigma_- - 2\sigma_-\rho_t^{(n-1)}\sigma_+\right)$$

- Splitting up $\rho_t = \sum_{n=0}^{\infty} \rho_t^{(n)}$, n photon emissions.

- Solution $G = \exp\{(\mathcal{L}_0 + sJ)t\}\rho(0)$, needs diagonalisation.
- In Laplace space, $\hat{G}(s,z) = [z - \mathcal{L}_0 - sJ]^{-1}\rho(0)$, needs Laplace inversion.
- $\hat{G}$ as vector, resolvent matrix

$$[z - \mathcal{L}_0 - sJ]^{-1} = \begin{pmatrix} z+2\beta & 0 & 0 & -\Omega \\ -2\beta s & z & 0 & \Omega \\ 0 & 0 & z+\beta & 0 \\ \frac{\Omega}{2} & -\frac{\Omega}{2} & 0 & z+\beta \end{pmatrix}.$$

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$$\dot{\rho}_t^{(n)} = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, \rho_t^{(n)}] - \beta\left(\sigma_+\sigma_-\rho_t^{(n)} + \rho_t^{(n)}\sigma_+\sigma_- - 2\sigma_-\rho_t^{(n-1)}\sigma_+\right)$$

- Splitting up $\rho_t = \sum_{n=0}^{\infty} \rho_t^{(n)}$, n photon emissions.

Result in Laplace space

$$\mathrm{Tr}\,\hat{G}(s,z) = \frac{(z+\beta)(z+2\beta) + \Omega^2 + (s-1)2\beta\left[(z+\beta)\rho_0^{++} + \Omega\mathrm{Im}\rho_0^{+-}\right]}{z(z+\beta)(z+2\beta) + \Omega^2[z+\beta(1-s)]}. \tag{18}$$

Resonance fluorescence: sub-Poissonian counting statistics

Information contained in

$$\mathrm{Tr}\hat{G}(s,z) = \frac{f(z)}{zf(z)+\beta\Omega^2(1-s)}, \quad f(z) \equiv (z+\beta)(z+2\beta)+\Omega^2.$$

Resonance fluorescence: sub-Poissonian counting statistics

Information contained in

$$\mathrm{Tr}\hat{G}(s,z) = \frac{f(z)}{zf(z)+\beta\Omega^2(1-s)}, \quad f(z) \equiv (z+\beta)(z+2\beta)+\Omega^2.$$

- Need to transform back into time-domain.

$$p_n(0,t) = \frac{\partial^n}{\partial s^n}\,\mathrm{Tr}G(s,t)|_{s=0}\,. \tag{19}$$

$$\langle n\rangle_t = \frac{\partial}{\partial s}\,\mathrm{Tr}G(s,t)|_{s=1} \quad \text{1st moment.} \tag{20}$$

$$\langle n(n-1)_t\rangle = \frac{\partial^2}{\partial s^2}\,\mathrm{Tr}G(s,t)|_{s=1} \quad \text{2nd factorial moment.} \tag{21}$$

Resonance fluorescence: sub-Poissonian counting statistics

Information contained in

$$\mathrm{Tr}\,\hat{G}(s,z) = \frac{f(z)}{zf(z)+\beta\Omega^2(1-s)}, \quad f(z) \equiv (z+\beta)(z+2\beta)+\Omega^2.$$

- Large t: pole z_0 closest to $z=0$.
- Expand $z_0 = \sum_{m=1}^{\infty} c_m (s-1)^m$

$$\begin{aligned} \rightsquigarrow \langle n \rangle_{t\to\infty} &= \frac{\beta\Omega^2}{2\beta^2+\Omega^2} t \\ \rightsquigarrow \sigma_t^2 \equiv \langle \Delta n^2 \rangle_{t\to\infty} &= \langle n \rangle_{t\to\infty} \left[1 - \frac{6\beta^2\Omega^2}{(2\beta^2+\Omega^2)^2} \right]. \end{aligned} \quad (19)$$

- Negative Mandel Q-parameter $Q \equiv F-1$, Fano factor $F \equiv \langle \Delta n^2 \rangle / \langle n \rangle < 1$.

Resonance fluorescence: sub-Poissonian counting statistics

Information contained in

$$\mathrm{Tr}\,\hat{G}(s,z) \;=\; \frac{f(z)}{zf(z)+\beta\Omega^2(1-s)}, \quad f(z)\equiv(z+\beta)(z+2\beta)+\Omega^2.$$

- Large $t \gg \beta^{-1}$: counting statistics $p_n(t)$ becomes a Gaussian!

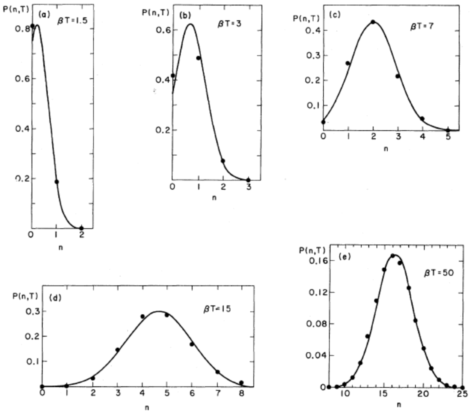
$$\lim_{t\to\infty} p_n(t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-(n-\bar{n}_t)^2/2\sigma_t^2}. \tag{19}$$

(D. Lenstra, PRA **26**, 3369 (1982)).

Resonance fluorescence: sub-Poissonian counting statistics

Information contained in

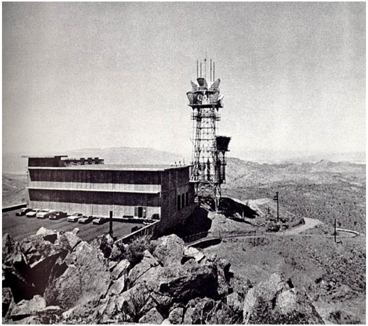
$$\mathrm{Tr}\hat{G}(s,z) \quad = \quad \frac{f(z)}{zf(z)+\beta\Omega^2(1-s)}, \qquad f(z) \equiv (z+\beta)(z+2\beta)+\Omega^2.$$



Counting in quantum optics: towards a counting formula...

Short revision

- Direct 'counting at the source': the savest option...
  - n-resolved master equations with 'jumpers' $J \to sJ$, generating operators. Cook 81 (Lesovik 89, Gurvitz 99, Bagrets/Nazarov 03 ...)
- Mandel (Poissonian) $p_n(t, t+T) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \quad \bar{n} = \eta I(\mathbf{r}) T$.
  - Classical field with intensity $I(\mathbf{r})$.
  - Golden rule (photo-electric effect) plus Markov.
- Mollow, Scully-Lamb single mode
  $p_n(0, t) = \mathrm{Tr}\rho(0) : \frac{1}{n!}\left(a^\dagger a \eta_t\right)^n \exp(-a^\dagger a \eta_t) :, \quad \eta_t \equiv 1 - e^{-\gamma t}$.
  - Correctly describes detector backaction. 'Closed system'.
  - Free cavity fields only, no sources.
- 'Quantum Mandel', Kelley-Kleiner $p_n(t, t+T) = \langle : \frac{\hat{\Omega}^n}{n!} e^{-\hat{\Omega}} : \rangle$.
  - Heisenberg operators, $\Omega \equiv \xi \int_t^{t+T} dt' \hat{E}^-(t')\hat{E}^+(t')$.
  - Not correct for long times. 'Open system'. Various generalisations on the market.



Three parties (source, field, detector).

Ueda's photodetector theory

M. Ueda PRA **41**, 3875 (1990). (Relatively) consistent attempt to put everything together ?

- Source-field interaction.
- Detector-field backaction.

Three parties (source, field, receiver/detector).

Multi-mode photodetector

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm D} + \mathcal{H}_{\rm FD}, \mathcal{H}_0 = \mathcal{H}_{\rm S} + \mathcal{H}_{\rm FS} + \mathcal{H}_{\rm F}$

$$\mathcal{H}_{\rm FD} = \sum_{Qkj} \left(V_k^Q c_{kj}^\dagger |0\rangle_j \langle 1| a_Q + H.c. \right), \quad \text{field-detector interaction} \quad (19)$$

Multi-mode photodetector

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm D} + \mathcal{H}_{\rm FD}, \mathcal{H}_0 = \mathcal{H}_{\rm S} + \mathcal{H}_{\rm FS} + \mathcal{H}_{\rm F}$

$$\mathcal{H}_{\rm FD} = \sum_{Qkj} \left(V_k^Q c_{kj}^\dagger |0\rangle_j \langle 1| a_Q + H.c. \right), \quad \text{field-detector interaction} \quad (19)$$

- Neglect $\mathcal{H}_{\rm FS}$ in deriving non-unitary part of master equation for χ_t (field-source density operator).

$$\begin{aligned} \frac{d}{dt}\chi_t^{(m)} &= -i[\mathcal{H}_0, \chi_t^{(m)}] \\ &- \frac{1}{2}\sum_{QQ'} \gamma_{QQ'} \left(a_Q^\dagger a_{Q'} \chi_t^{(m)} + \chi_t^{(m)} a_Q^\dagger a_{Q'} - 2 a_{Q'} \chi_t^{(m-1)} a_Q^\dagger \right). \end{aligned} \quad (20)$$

- Assumes 'broadband detection', $\gamma_{QQ'} = 2\pi N \sum_k V_k^Q \bar{V}_k^{Q'} \delta(\varepsilon - \varepsilon_{kj})$, $N \gg m$ detector atoms.

Formal solution

Generating operator G, 'damper' $\mathcal{L}_0$, 'jumper' J.

- Write $\partial_t G = \mathcal{L}_0 G + sJG$, $G(s,t) \equiv \sum_{m=0}^{\infty} s^m \chi_t^{(m)}$.
- $\mathcal{L}_0 X \equiv YX + XY^\dagger$, $Y \equiv -i\mathcal{H}_0 - \frac{1}{2}\sum_{QQ'} \gamma_{QQ'} a_Q^\dagger a_{Q'}$.
- $JX \equiv \sum_{QQ'} \gamma_{QQ'} a_{Q'} X a_Q^\dagger$.

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- Interaction picture $G(s,t) \equiv S_t \tilde{G}(s,t)$, $S_t \equiv e^{\mathcal{L}_0 t}$.
- Here, $S_t X \equiv e^{\mathcal{L}_0 t} X = e^{Yt} X e^{Y^\dagger t}$.
- Counting and jumping in interaction picture,

$$\partial_t \tilde{G}(s,t) = s e^{-\mathcal{L}_0 t} J e^{\mathcal{L}_0 t} \tilde{G}(s,t). \tag{21}$$

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Solution of $\partial_t \tilde{G}(s,t) = se^{-\mathcal{L}_0 t} J e^{\mathcal{L}_0 t} \tilde{G}(s,t)$ as formal power series,

$$\begin{aligned}
\tilde{G}(s,t) &= \tilde{G}(s,0) + \int_0^t dt' s e^{-\mathcal{L}_0 t'} J e^{\mathcal{L}_0 t'} \left\{ \tilde{G}(s,0) + \int_0^{t'} dt'' s \ldots \right\} \\
&= \sum_{m=0}^{\infty} s^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 S_{-t_m} J S_{t_m - t_{m-1}} J \ldots J S_{t_m} \chi(0) \\
G(s,t) &= \sum_{m=0}^{\infty} s^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 S_{t-t_m} J S_{t_m - t_{m-1}} J \ldots J S_{t_m} \chi(0). \quad (21)
\end{aligned}$$

Formal solution

Generating operator G, 'damper' $\mathcal{L}_0$, 'jumper' J.

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- $JX \equiv \sum_{QQ'} \gamma_{QQ'} a_{Q'} X a_Q^\dagger$.

Single-mode case first for simplicity ($A(t) \equiv e^{-Yt} a e^{Yt}$):

$$
\begin{aligned}
\tilde{\rho}_t^{(m)} &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 A(t_m)...A(t_1)\chi(0)A^\dagger(t_1)...A^\dagger(t_m) \\
\rho_t^{(m)} &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 e^{Yt} A(t_m)...A(t_1)\chi(0)A^\dagger(t_1)...A^\dagger(t_m) e^{Y^\dagger t}.
\end{aligned}
$$

Formal solution

Generating operator G, 'damper' $\mathcal{L}_0$, 'jumper' J.

- Write $\partial_t G = \mathcal{L}_0 G + sJG$, $G(s,t) \equiv \sum_{m=0}^{\infty} s^m \chi_t^{(m)}$.
- $\mathcal{L}_0 X \equiv YX + XY^\dagger$, $Y \equiv -i\mathcal{H}_0 - \frac{1}{2}\sum_{QQ'} \gamma_{QQ'} a_Q^\dagger a_{Q'}$.
- $JX \equiv \sum_{QQ'} \gamma_{QQ'} a_{Q'} X a_Q^\dagger$.

Single mode case, taking traces:

$$
\begin{aligned}
\mathrm{Tr}\tilde{\rho}_t^{(m)} &= \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle A^\dagger(t_1)\ldots A^\dagger(t_m) A(t_m)\ldots A(t_1)\rangle \\
\mathrm{Tr}\rho_t^{(m)} &= \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle A^\dagger(t_1)\ldots A^\dagger(t_m) e^{Y^\dagger t} e^{Yt} A(t_m)\ldots A(t_1)\rangle .
\end{aligned}
$$

Relation with Kelley-Kleiner formula

Ueda vs Kelley-Kleiner

$$\begin{aligned} p_m^{\mathrm{U}}(t) &= \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle A^\dagger(t_1)\ldots A^\dagger(t_m) e^{Y^\dagger t} e^{Yt} A(t_m)\ldots A(t_1)\rangle \\ p_m^{\mathrm{KK}}(t) &= \langle : \frac{\hat{\Omega}^m}{m!} e^{-\hat{\Omega}} : \rangle, \quad \hat{\Omega} \equiv \xi \int_0^t dt' a^\dagger(t') a(t') \end{aligned} \tag{21}$$

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$$p_m^{\mathrm{U}}(t) = \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle A^\dagger(t_1)\ldots A^\dagger(t_m) e^{Y^\dagger t} e^{Yt} A(t_m)\ldots A(t_1)\rangle$$

$$p_m^{\mathrm{KK}}(t) = \langle : \frac{\hat{\Omega}^m}{m!} e^{-\hat{\Omega}} : \rangle, \quad \hat{\Omega} \equiv \xi \int_0^t dt' a^\dagger(t') a(t') \tag{21}$$

- No detector backaction in KK.
- Replace damped time-evolution $A(t) \equiv e^{-Yt} a e^{Yt}$ by free time-evolution $a(t) \equiv e^{i\mathcal{H}_0 t} a e^{-i\mathcal{H}_0 t}$.
- Remember single mode case (Mollow, Scully-Lamb) $p_m(t) = \mathrm{Tr}\left\{\rho(0) : \frac{1}{m!}\left(a^\dagger a \eta_t\right)^m \exp(-a^\dagger a \eta_t) :\right\}$, $\eta_t \equiv 1 - e^{-\gamma t}$.
- KK is short-time limit $\gamma t \ll 1 \rightsquigarrow \eta_t = \gamma t$.

Relation with Kelley-Kleiner formula

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$$\begin{aligned} p_m^{\mathrm{U}}(t) &= \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle A^\dagger(t_1)\ldots A^\dagger(t_m) e^{Y^\dagger t} e^{Yt} A(t_m)\ldots A(t_1)\rangle \\ p_m^{\mathrm{KK}}(t) &= \langle : \frac{\hat{\Omega}^m}{m!} e^{-\hat{\Omega}} : \rangle, \quad \hat{\Omega} \equiv \xi \int_0^t dt' a^\dagger(t') a(t') \end{aligned} \tag{21}$$

Up to first order in γ

$$\begin{aligned} e^{Y^\dagger t} e^{Yt} &= \left(1 + \frac{\gamma}{2}\int_0^t dt' a^\dagger(t') a(t') \ldots\right)\left(1 + \frac{\gamma}{2}\int_0^t dt' a^\dagger(t') a(t') \ldots\right) \\ &= \left(1 + \gamma \int_0^t dt' a^\dagger(t') a(t') \ldots\right) \\ &= : \exp\left(\gamma \int_0^t dt' a^\dagger(t') a(t')\right) : \end{aligned} \tag{22}$$

Relation with Kelley-Kleiner formula

Ueda vs Kelley-Kleiner

$$
\begin{aligned}
p_m^{\mathrm{U}}(t) &= \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle A^\dagger(t_1)\ldots A^\dagger(t_m) e^{Y^\dagger t} e^{Yt} A(t_m)\ldots A(t_1)\rangle \\
p_m^{\mathrm{KK}}(t) &= \langle : \frac{\hat{\Omega}^m}{m!} e^{-\hat{\Omega}} : \rangle, \quad \hat{\Omega} \equiv \xi \int_0^t dt' a^\dagger(t') a(t') \qquad (21)
\end{aligned}
$$

- Sum-rule $\sum_{m=0}^{\infty} p_m(0,t) = 0$ fulfilled for

$$
\begin{aligned}
p_m(0,t) &\equiv \mathrm{Tr}\rho_t^{(m)} = \qquad (22) \\
&= \gamma^m \int_0^t dt_m \ldots \int_0^{t_2} dt_1 \langle : a^\dagger(t_1) a(t_1) \ldots a^\dagger(t_m) a(t_m) e^{\gamma \int_0^t dt' a^\dagger(t') a(t')} : \rangle \\
&= \langle : \frac{1}{m!} \left[\gamma \int_0^t dt' a^\dagger(t') a(t') \right]^m e^{\gamma \int_0^t dt' a^\dagger(t') a(t')} : \rangle .
\end{aligned}
$$

Multi-mode form

$$p_m(0,t) \equiv \mathrm{Tr}\rho_t^{(m)} = \sum_{Q_1 Q_1' \ldots Q_m Q_m'} \gamma_{Q_1 Q_1'} \ldots \gamma_{Q_m Q_m'} \times$$

$$\times \int_0^t dt_m \ldots \int_0^{t_1} dt_1 \mathrm{Tr}\left(\chi_0 a^\dagger_{Q_1}(t_1) \ldots a^\dagger_{Q_m}(t_m) e^{Y^\dagger t} e^{Yt} a_{Q_m'}(t_m) \ldots a_{Q_1'}(t_1)\right).$$

- Somewhat impractical ...
- Counting-at-source method much simpler.
- Alternative: integrate out fields in $\partial_t G = \mathcal{L}_0 G + sJG$ (?)

Quantum trajectories: this should now be easy...

Quantum jump method, Monte-Carlo for master equation

Example: spontaneous emission from TLS (rotating frame)

$$\dot{\rho}_t = -\beta\left(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+\right)$$

Quantum jump method, Monte-Carlo for master equation

Example: spontaneous emission from TLS (rotating frame)

$\dot{\rho}_t = -\beta \left(\sigma_+ \sigma_- \rho_t + \rho_t \sigma_+ \sigma_- - 2\sigma_- \rho_t \sigma_+ \right)$

- Jump super-operator J with $J\rho = 2\beta\sigma_- \rho \sigma_+$
- Solve $\partial_t \rho_t = (\mathcal{L}_0 + J)\rho_t$.
- Interaction picture with respect to $\mathcal{L}_0$: $\rho_t \equiv S_t \tilde{\rho}_t$, $S_t \equiv e^{\mathcal{L}_0 t}$.

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- Interaction picture with respect to $\mathcal{L}_0$: $\rho_t \equiv S_t\tilde{\rho}_t$, $S_t \equiv e^{\mathcal{L}_0 t}$.
- Solution of $\partial_t\tilde{\rho}(t) = e^{-\mathcal{L}_0 t} J e^{\mathcal{L}_0 t}\tilde{\rho}(t)$ as formal power series,

$$\rho(t) = \sum_{m=0}^{\infty}\int_0^t dt_m \dots \int_0^{t_2} dt_1 \underline{S_{t-t_m} J S_{t_m - t_{m-1}} J \dots J S_{t_1}\rho(0)}. \quad (23)$$

- m quantum jumps occuring at times $t_1, \dots, t_m$.
- Sum over all 'trajectories' with $m = 0, 1, \dots, \infty$ jumps between 'free' (but damped) time-evolution.

Quantum jump method, Monte-Carlo for master equation

Example: spontaneous emission from TLS (rotating frame)

$\dot{\rho}_t = -\beta\left(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+\right)$

Monte-Carlo procedure. Fixed time step Δt.

- Step 1: start with pure *wave function* $|\Psi\rangle$.
- Step 2: calculate collaps probability, $P_{\rm col} = \beta\Delta t\langle\Psi|\sigma_+\sigma_-|\Psi\rangle$
- Step 3: compare $P_{\rm col}$ with random number $0 \leq r \leq 1$.
  - If $P_{\rm col} > r$, replace $|\Psi\rangle \rightarrow \sigma_-|\Psi\rangle/\|\sigma_-|\Psi\rangle\|$.
  - If $P_{\rm col} \leq r$, no emission but time-evolution $|\Psi\rangle \rightarrow (1 - i\Delta t H_{\rm eff}|\Psi\rangle/\mathcal{N}$, where $H_{\rm eff}) = -i\beta\sigma_+\sigma_-$.
- Go back to Step 2.
- Repeat procedure in order to obtain average.

Quantum jump method, Monte-Carlo for master equation

Example: spontaneous emission from TLS (rotating frame)

$\dot{\rho}_t = -\beta\left(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+\right)$

Monte-Carlo procedure. Fixed time step Δt.

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- Repeat procedure in order to obtain average.

- Widely used in quantum optics community.
- Note: splitting $\mathcal{L} = \mathcal{L}_0 + J$ is not unique.
- Literature: Carmichael (book); Plenio,Knight (review).

Summary

- Multi-mode quantum optics: field as 'bath'.
- Correlation (coherence) functions.
- Resonance fluorescence: 'counting at the source', sub-Poissonian, anti-bunched.
- Multi-mode photo-detector theory.
- Quantum trajectories.

Summary

- Multi-mode quantum optics: field as 'bath'.
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- Multi-mode photo-detector theory.
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Still to do

- Microscopic models for source-field-detector.
- Further understanding of counting statistics $p_n(t)$.
- More complex quantities, e.g. time-resolved probabilities $P_n(t_1, ..., t_n; [t, t+T])$.