An introduction to

Full Counting Statistics

in Mesoscopic Electronics

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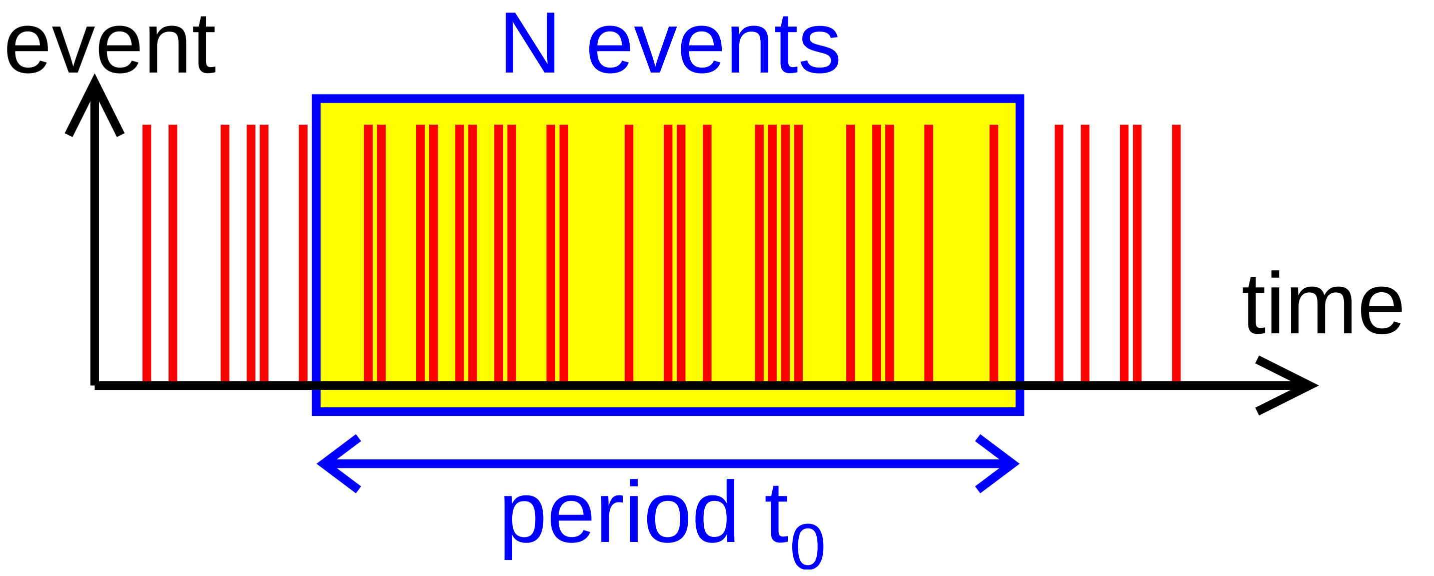
Full Counting Statistics: Introduction

Distribution of events
(classically occuring in certain time interval t_0)



Examples for countable events:

1. trains arriving in a station
2. the occurrence of 0s in roulette games
3. number of electrons in electric current

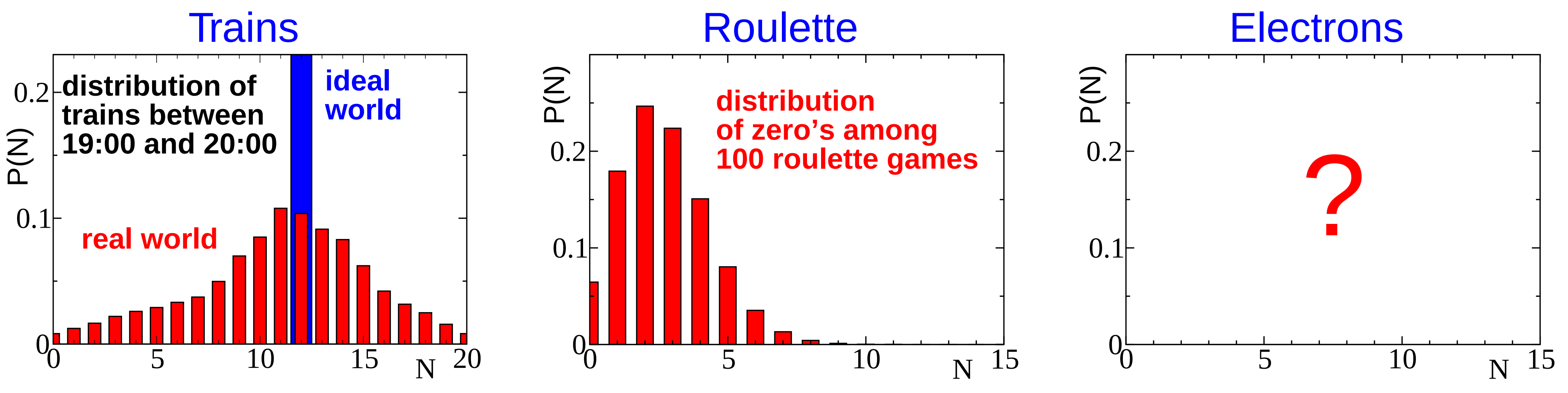
Question: How to characterize the distribution of events?

Full Counting Statistics: Introduction

Averages (by repeated experiments/observations)

- mean number of events $\overline{N}$
- variance of number $\overline{(N - \overline{N})^2}$
- however: much information disregarded!

Complete characterization of events: number distributions



$P(N)$: Probabilty to observe N events

Full Counting Statistics: Introduction

Measuring electric current = counting electrons

Average current measurement: $It_0 = \langle N \rangle = \overline{N}$
Individual measurement gives not necessarily $\overline{N}$, but some (integer) number N
Described by probability distribution $P(N)$

Fundamental question:

What is the statistics of the transfered charge N?

Full Counting Statistics $P(N)$

Information on

- what is the elementary charge transfer
- statistics of particles (e.g. fermions/bosons/uncorrelated)
- correlations of two and more particles
- mesoscopic PIN-code (all transmission eigenvalues)

Full Counting Statistics: Introduction

Probability theory: Probability distribution: $P(N)$

Normalization: $\sum_N P(N) = 1 = M_0$

Average of N: $\langle N \rangle = \overline{N} = \sum_N N P(N)$

General moments: $M_n = \langle N^n \rangle$

Central moments: $\overline{M}_n = \left\langle \left(N - \overline{N}\right)^n \right\rangle$

Definition: moment generating function

$$\Phi(\chi) = \left\langle e^{iN\chi} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (i\chi)^n M_n$$

$$\rightarrow M_n = \left(-i \frac{\partial}{\partial \chi}\right)^n \Phi(\chi) \Big|_{\chi \to 0}$$

Normalization: $\Phi(0) = 1$

Full Counting Statistics: Introduction

Definition: cumulant generating function **(CGF)**

$$\begin{aligned} S(\chi) &= \ln \Phi(\chi) \\ e^{S(\chi)} &= \langle e^{iN\chi} \rangle = \sum_N e^{iN\chi} P(N) \end{aligned}$$

Expansion (defines cumulants): $S(\chi) = \sum_{n=1}^{\infty} \frac{1}{n!} (i\chi)^n C_n$

Relation cumulants ↔moments

$$\begin{aligned} C_1 &= M_1 \\ C_2 &= \overline{M}_2 = \langle N^2 \rangle - \langle N \rangle^2 \\ C_3 &= \overline{M}_3 = \langle \left(N - \overline{N}\right)^3 \rangle \\ C_4 &= \overline{M}_4 - 3\overline{M}_2{}^2 \end{aligned}$$

Normalization: $S(0) = 0$

Full Counting Statistics: Introduction

Multivariate distributions :

Joint probability for K different events: $P(N_1, N_2, \ldots, N_K) \equiv P(\vec{N})$
correspondingly $\chi \to \vec{\chi} = (\chi_1, \chi_2 \ldots, \chi_K)$

Cumulant generating function: $e^{S(\vec{\chi})} = \left\langle e^{i\vec{N}\vec{\chi}} \right\rangle$

Correlations: $C_{ij} = \langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle$

Example: independent events $P(N_1, N_2) = P_1(N_1) P_2(N_2)$

Thus $S(\chi_1, \chi_2) = S_1(\chi_1) + S_2(\chi_2)$
$\longrightarrow$CGF's of independent events are additive

Consequence: $C_{12} = \left\langle (N_1 - \overline{N}_1)(N_2 - \overline{N}_2) \right\rangle = -\frac{\partial}{\partial \chi_1} \frac{\partial}{\partial \chi_2} S(\vec{\chi}) \Big|_{\vec{\chi}=0}$

$= 0$ if 1 and 2 are independent

Reverse: if CGF can be written as sum $\longrightarrow$
terms can be interpreted as independent events

Full Counting Statistics: Introduction

Some well known probability distributions

	$P(N)$	$S(\chi)$
delta-distribution	$\delta_{N,\overline{N}}$	$i\overline{N}\chi$
Gauss	$\frac{1}{\sqrt{2\pi\sigma}} e^{\frac{(N-\overline{N})^2}{2\sigma}}$	$i\overline{N}\chi - \sigma\chi^2/2$
Poisson	$\frac{\overline{N}^N}{N!} e^{-\overline{N}}$	$\overline{N}(e^{i\chi} - 1)$
Binomial	$\binom{M}{N} p^N (1-p)^{M-N}$	$M \ln\left[1 + p\left(e^{i\chi} - 1\right)\right]$

Multinomial distribution

$$P(\vec{N}) = \frac{M!}{N_1!N_2!\cdots N_K!(M-\sum_i N_i)} p_1^{N_1} p_2^{N_2} \cdots p_K^{N_K} \left(1 - \sum_i p_i\right)^{M-\sum_i N_i}$$

$$S(\vec{\chi}) = M \ln\left[1 + \sum_{n=1}^{K} p_n \left(e^{i\chi_n} - 1\right)\right]$$

Electron Counting Statistics: Quantum Theory

Formal classical definition of FCS:

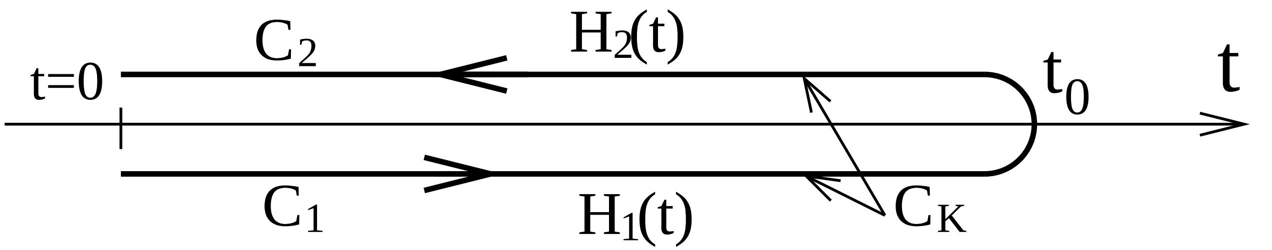
$$P(N) = \left\langle \delta(N - \hat{N}) \right\rangle$$

Charge operator: $\hat{N} = \int_0^{t_0} dt \hat{I}(t)$

Cumulant generating function $e^{S(\chi)} = \left\langle e^{i\chi\hat{N}} \right\rangle$

<u>Quantum mechanical definition</u>

Keldysh time-contour:



Characteristic function $\Phi(\chi) = \left\langle \mathcal{T}_K e^{-\frac{i}{2}\int_{C_K} dt \chi(t) \hat{I}(t)} \right\rangle$

Electrons on upper and lower branch see different Hamiltonians

$H_{1(2)}(t) = H_0 \pm \chi \hat{I}$ obtained by $\chi(t) = \begin{cases} +\chi & t \in C_1 \\ -\chi & t \in C_2 \end{cases}$.

Full Counting Statistics: Theory

Motivation for quantum definition of CGF (rough sketch)

Charge detector density matrix (in Wigner representation)

$$\rho(\phi, Q, t_0) = \sum_N P(\phi, N, t_0)\rho(\phi, Q - N, 0)$$

$\phi =$ conjugate variable of Q

"Probability" determines the time evolution of the detector density matrix

$$P(\phi, N, t_0) = \int d\chi e^{iN\chi + S(\phi+\chi/2, \phi-\chi/2, t_0)}$$

where

$$e^{S(\chi_1, \chi_2, t_0)} = \text{Tr}_{\text{system}} \mathcal{T} e^{i\chi_1 \int_0^{t_0} dt I(t)} \rho_0 \tilde{\mathcal{T}} e^{-i\chi_2 \int_0^{t_0} dt I(t)}$$

If $S(\chi_1, \chi_2, t_0)$ depends only on difference $\chi = \chi_1 - \chi_2$ $\rightarrow P(N, t_0)$ independent on ϕ $\rightarrow$probability of charge transfer

[Nazarov and Kindermann, Eur. Phys. J. B 03]

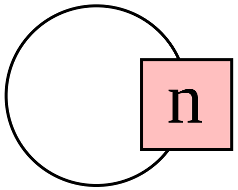
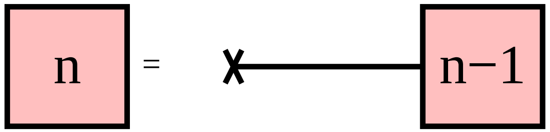
Full Counting Statistics: Theory

Use standard Green's function methods with time-dependent Hamiltonian.

'Cumulant expansion'

$$e^{S(\chi)} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\chi)^n \left\langle \mathcal{T}_K \hat{N}^n \right\rangle$$
$$= \exp\left[\sum_{n=1}^{\infty} \frac{(i\chi)^n}{n} \left\langle \mathcal{T}_K \hat{N}^n \right\rangle_{\text{connected}}\right]$$

It follows that:

- $S(\chi) = \sum_{n=1}^{\infty} \frac{1}{n} \left(i\frac{\chi}{2}\right)^n$  ,
- where 
- Vertex: $\check{N} = \check{\tau}_K \int_0^{t_0} dt \hat{I}$ in Keldysh matrix space
- Line: single particle Green's function $\check{G}$
- Current operator matrix $\check{\tau}_K = \overline{\tau}_z$ (corresponds to $\pm\chi$ for lower/upper contour) other matrix structures included in $\hat{I}$

Full Counting Statistics: Theoretical Approach

Definition of χ-dependent Green's function:

$$\left[i\frac{\partial}{\partial t} - \hat{H}_0 - \frac{\chi}{2}\check{\tau}_K\hat{I}\right]\check{G}(\chi, t, t') = \delta(t - t')$$

'Charge' operator is a matrix in Keldysh-space: $\check{N} = \check{\tau}_K \int_0^{t_0} dt\hat{I}$

The 'unperturbed' Green's function is

$$\left[i\frac{\partial}{\partial t} - \hat{H}_0\right]\check{G}_0 = \check{1}$$

(written in obvious matrix-notation)

Pertubation expansion for Green's function $\check{G}(\chi)$ yields

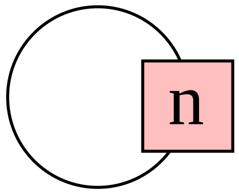
$$\check{G}(\chi) = \sum_{n=1}^{\infty}\left(i\frac{\chi}{2}\right)^{n-1}\check{G}_0\left(\check{N}\check{G}_0\right)^{n-1}$$

Full Counting Statistics: Theoretical approach

We define the χ-dependent current via

$$t_0 I(\chi)/e = \sum_{n=1}^{\infty} \left(i\frac{\chi}{2}\right)^{n-1} \mathrm{Tr}\left(\check{N}\check{G}_0\right)^n$$

$$= \sum_{n=1}^{\infty} 1 \left(i\frac{\chi}{2}\right)^{n-1}$$



Comparison of expansion for $S(\chi)$ and $I(\chi)$ leads to relation of Green's function formalism to counting statistics [Nazarov 1999]

$$I(\chi) = -i\frac{e}{t_0}\frac{\partial S(\chi)}{\partial \chi}$$

allows to use (in principle...) all techniques for GF

Quasiclassical Approximation

Quasiclassical approximation

Green's functions in real space oscillate on length scale of Fermi wavelength λ_F:

$$\check{G}(x, x') \sim e^{ik_F(x-x')} g(x, x')$$

$g(x, x')$ is the envelope function

The relevant length scales in many systems are

- l_{imp} elastic mean free path
- $\xi_T = \hbar v_F / k_B T$ temperature coherence length
- $\xi_S = \hbar v_F / \Delta$ superconducting coherence length
- etc.

In the limit $\lambda_F \ll l_{imp}, \xi_T, \xi_S$

quasiclassical approximation
(theory for envelope functions only!)

Quasiclassical Approximation

Fast oscillations of Green's functions are integrated out
$\longrightarrow$quasiclassical Green's functions

$$\check{g}(\vec{r}, \vec{v}_F)$$

$\vec{r} = \frac{1}{2}(\vec{x} + \vec{x}')$ center of mass coordinate
$\vec{v}_F$ direction on Fermi surface

Effective equation of motion $\longrightarrow$Eilenberger equation

$$-i\hbar\vec{v}_F\nabla\check{g} = \left[i\check{h}_0 + i\check{\sigma}, \check{g}\right]$$

$\check{h}_0 =$ 'rest Hamiltonian' ; $\check{\sigma} =$ self-energy

- much simpler equation
- homogenous $\longrightarrow$bulk solutions have to be supplied
- invalid near interfaces $\longrightarrow$extra boundary conditions necessary

Normalization condition: $\check{g}^2 = \check{1}$

Quasiclassical Approximation

Bulk solutions for Keldysh-Green's functions : $\check{G} = \begin{pmatrix} \hat{R} & \hat{K} \\ 0 & \hat{A} \end{pmatrix}$; $\check{G}^2 = \check{1}$

$\hat{R}, \hat{A}$: spectral properties (Retarded, Advanced)

$\hat{K}$: occupation of spectrum (Keldysh)

Nambu substructure: $\hat{R}, \hat{A}, \hat{K} = \begin{pmatrix} G_{R,A,K} & F_{R,A,K} \\ -F_{R,A,K} & -G_{R,A,K} \end{pmatrix}$

G : normal Green's function (density of states, distribution function)

F : anomalous Green's function (pair correlations)

Normal metal

$\hat{R} = -\hat{A} = \hat{\tau}_3$

$\hat{K} = \begin{pmatrix} 1 - 2f(E) & \\ & 2f(-E) - 1) \end{pmatrix}$

$\hat{\tau}_i$ = Pauli matrices

Superconductor (in equilibrium)

$\hat{R}(\hat{A}) = (E\hat{\tau}_3 - i\Delta\hat{\tau}_2) / \Omega_{R(A)}$

$\Omega^2_{R(A)} = (E \pm i0)^2 - |\Delta|^2$

$\hat{K} = \left(\hat{R} - \hat{A}\right) \tanh(E/2k_BT)$

Quasiclassical Approximation

Examples for selfenergies:

- $\hat{\sigma}_{imp} = \frac{\hbar}{2\tau_{imp}} \langle \check{g} \rangle_{v_F}$ elastic impurity scattering
- $\check{\Delta}$ superconducting gap-matrix
- electron-phonon scattering, etc.

$\langle \ldots \rangle_{v_F} = \int d\Omega_{v_F} / 4\pi$ average over the Fermi surface

Quasiclassical current:

$$\vec{j} = \frac{e\nu}{4} \int dE \, \mathrm{Tr} \langle \vec{v}_F \check{\tau}_K \check{g} \rangle_{v_F}$$

ν : density of states at the Fermi energy

Quasiclassical Approximation: Diffusive Limit

Disordered system: $\lambda_F \ll l_{imp} \ll \xi_T, \xi_\Delta, \ldots$

Dominating term in Eilenberger equation: impurity selfenergy $\frac{1}{2\tau_{imp}} \langle \check{g} \rangle_{v_F}$

Green's function are (almost) isotropic

Ansatz: $\check{g}(x, \vec{v}_F) \approx \check{G}(x) + \vec{v}_F \check{\vec{g}}_1(x)$, $\vec{v}_F \check{\vec{g}}_1(x) \ll \check{G}(x)$

Eilenberger $\longrightarrow$'Usadel'-equation

$$\begin{aligned}
\nabla D(\mathrm{x}) \check{G}(\mathrm{x}) \nabla \check{G}(\mathrm{x}) &= \left[-iE\check{\sigma}_3, \check{G}(\mathrm{x})\right] \\
\check{\mathrm{I}}(\mathrm{x}) &= -\sigma(\mathrm{x}) \check{G}(\mathrm{x}) \nabla \check{G}(\mathrm{x}) \\
\vec{j} &= \frac{1}{4e} \int dE \, \mathrm{tr} \check{\tau}_K \check{I}
\end{aligned}$$

Matrix Diffusion Equation

Quasiclassics and Counting Statistics

Inclusion of full counting statistics ($H(\chi) = H_0 + i\frac{\chi}{2}\check{\tau}_K I_{op}$):

Counting charge somewhere inside terminal

Current through some cross section C

Definition: $F(x)$, which changes from 0 to 1 at C
on length scale $\Lambda \ll l_{imp}, \xi_T, \ldots$

Current operator on C: $\check{j}_C(x) = \vec{v}_F \check{\tau}_K \left(\nabla F(x)\right)$

Eilenberger equation in the vicinity of C: $-i\vec{v}_F \nabla \check{g} = \left[\frac{\chi}{2}\vec{v}_F \check{\tau}_K \left(\nabla F(x)\right), \check{g}\right]$

Solution ('gauge transformation'): $\check{G}(x) = e^{i\frac{\chi}{2}\check{\tau}_K F(x)} \check{G}_0 e^{-i\frac{\chi}{2}\check{\tau}_K F(x)}$

New boundary condition: $\check{G}(\chi) = e^{i\frac{\chi}{2}\check{\tau}_K} \check{G}_0 e^{-i\frac{\chi}{2}\check{\tau}_K}$

$\check{G}_0 =$ Green's function in the absence of counting (e.g. $\check{G}_N, \check{G}_S$)

Full Counting Statistics: Summary Theoretical Approach

FCS defined in terms of extended Keldysh-Green's function formalism

Approach

- New terminal Green's functions

$$\check{G}(\chi) = e^{i\frac{\chi}{2}\check{\tau}_K}\check{G}_0 e^{-i\frac{\chi}{2}\check{\tau}_K}$$

  Remark: these Green's functions merely determine the boundary condition at ∞ (a la Landauer), they are always quasiclassical

- proceed 'as usual' to find the average current (but respecting in all steps the full matrix structure, i.e. the dependence the counting field χ)
- the CGF is obtained via the relation

$$I(\chi) = -i\frac{e}{t_0}\frac{\partial S(\chi)}{\partial \chi}$$

- all correlation functions determined

Tunnel Junction

Hamiltonian: $H = H_L + H_R + H_T$

Perturbation expansion in H_T (to second order)

Result for the current (G_T tunnel conductance)

$$I(\chi) = \frac{G_T}{8e} \int dE \, \mathrm{Tr} \left(\check{\tau}_K \left[\check{G}_L(\chi), \check{G}_R \right] \right) \, .$$

The CGF is (using $\frac{\partial}{\partial \chi} G_L(\chi) = \frac{i}{2} \left[\check{\tau}_K, \check{G}_L(\chi) \right]$)

$$S(\chi) = i \frac{t_0}{e} \int_0^{\chi} d\chi' I(\chi') = \frac{G_T t_0}{4e^2} \int dE \, \mathrm{Tr} \left\{ \check{G}_L(\chi), \check{G}_R \right\}$$

Tunnel Junction

Extracting the dependence on the counting field

$$S(\chi) = N_R(e^{i\chi} - 1) + N_L(e^{-i\chi} - 1)\,.$$

- $N_{L(R)} = \frac{t_0 G_T}{4e^2} \int dE \,\mathrm{Tr}\left[(1 \pm \check{\tau}_K)\check{G}_L(1 \mp \check{\tau}_K)\check{G}_R\right]$
- FCS = bidirectional Poisson distribution
- $N_{R(L)}$ are the average numbers of charges that are tunneling to the right (left)
- holds for normal and superconducting junctions
- cumulants $C_{2n+1} = N_R - N_L \quad , \quad C_{2n} = N_R + N_L$
- tunneling only in one direction (N_L or N_R vanish)

  Statistics is Poisson: $P(N) = \frac{\overline{N}^N}{N!} e^{-\overline{N}}$ (e.g. third cumulant $C_3 = e^2 \overline{N}$)
- in equilibrium $N_L = N_R$: non-Gaussian statistics

Caution: here the Keldysh time-ordering is essential. “Classical” CGF gives wrong result (e.g. third cumulant $\left\langle \left[\int_0^{t_0} dt I(t) - e\overline{N} \right]^3 \right\rangle = 0!$)

Two Terminal Contact

General contact (described by scattering matrix):

important quantities $\{T_n\}$ (transmission matrix eigenvalues)

Current is given by [Nazarov, 1999]

$$\check{I} = -\frac{e^2}{\pi}\sum_n \frac{T_n\left[\check{G}_L, \check{G}_R\right]}{4 + T_n\left(\{\check{G}_L, \check{G}_R\} - 2\right)}$$

Counting statistics obtained from

$$S(\chi) = i\frac{t_0}{e}\int_0^\chi d\chi' I(\chi') \quad ; \quad I(\chi) = \frac{1}{4e}\int dE \,\mathrm{tr}\left[\check{\tau}_K \check{I}\right]$$

Result: $\mathrm{Tr} = \sum_n \int dE\ \mathrm{tr}$; tr = trace in Keldysh-Nambu-... space

$$S(\chi) = \frac{t_0}{4\pi}\mathrm{Tr}\ln\left[1 + \frac{T_n}{4}\left(\{\check{G}_L(\chi), \check{G}_R\} - 2\right)\right]$$

→ Full counting statistics of mesoscopic two terminal contact

[W.B. and Yu.V. Nazarov, PRL 2001]

Single-Channel Contact

One channel contact (T_1) between two normal metals (occupations $f_{L,R}$):

[Lesovik and Levitov, JETPL 1993]

$$S(\chi) = \frac{t_0}{\pi} \int dE \ln \left[1 + \underbrace{T_1 f_L (1 - f_R)}_{\text{L} \longrightarrow \text{R transfer}} \underbrace{\left(e^{i\chi} - 1 \right)}_{\text{counting factor}} + \underbrace{T_1 f_R (1 - f_L)}_{\text{R} \longrightarrow \text{L transfer}} \underbrace{\left(e^{-i\chi} - 1 \right)}_{\text{counting factor}} \right]$$

At zero temperature and bias V:

$$S(\chi) = \underbrace{\frac{t_0 eV}{\pi}}_{M} \ln \left[1 + T_1 \left(e^{i\chi} - 1 \right) \right]$$

- Binomial statistics: $P(N) = \binom{M}{N} T_1^N (1 - T_1)^{M-N}$ with $M = \frac{t_0 eV}{\pi}$ # of attempts
- Anticorrelated transport of electrons (second cumulant $C_2 = MT(1 - T)$)
- third cumulant: $C_3 = MT(1 - T)(1 - 2T)$

Single-Channel Contact

Origin of the trinomial statistics:

electrons in leads (n_L, n_R)	occupations	scattering	charge q
(0,0)	$(1-f_L)(1-f_R)$	1	0
(1,0)	$f_L(1-f_R)$	T	+1
		R	0
(0,1)	$f_R(1-f_L)$	T	-1
		R	0
(1,1)	$f_R f_L$	1	0
		Pauli!	

The CGF for a single scattering process

$$\begin{aligned} e^{S_E(\chi)} &= \sum_{n_L,n_R} p(n_L,n_R) e^{i\chi q} \\ &= 1 + T f_L(1-f_R)\left(e^{i\chi}-1\right) + T f_R(1-f_L)\left(e^{-i\chi}-1\right) \end{aligned}$$

CGFs for all scattering processes add up coherently!

$$S(\chi) = \sum_{\text{spin}} \frac{t_0}{h} \int dE\, S_E(\chi)$$ →Levitov formula

Single-Channel Andreev Contact

One channel contact (T_1) between normal- and superconductor :

[Muzykantskii and Khmelnitskii, PRB 1994]

$$S(\chi) = \frac{t_0}{2\pi} \int dE \ln \left[1 + \sum_{\sigma=\pm} \underbrace{A_{\sigma 2} \left(e^{\sigma i 2\chi} - 1 \right)}_{\text{2-particle transfer}} + \underbrace{A_{\sigma 1} \left(e^{\sigma i \chi} - 1 \right)}_{\text{1-particle transfer}} \right]$$

At $T = 0$ and bias $eV \ll \Delta$:

$$S(\chi) = M \ln \left[1 + \underbrace{R_A} \left(e^{i2\chi} - 1 \right) \right]$$

$$R_A = T_1^2/(2 - T_1)^2$$

probability of Andreev reflection

Statistics is binomial:

from $S(\chi + \pi) = S(\chi)$ follows

$$P(N = 2n + 1) = 0$$

$$P(N = 2n) = \binom{M}{n} R_A^n (1 - R_A)^{M-n}$$

$\longrightarrow$ vanishes for odd N

Interpretation:

- $M =$ same as NN-contact
- Andreev reflection
  $\rightarrow$doubled charge transfer
- Noise
  $C_2 = 4MR_A(1 - R_A)$

Single-Channel Andreev Contact

Uncorrelated two-particle scattering: (for simplicity zero temperature)

$$
\begin{aligned}
S_2 &= \sum_\sigma \ln\left[1+T(e^{i\chi}-1)\right] \\
&= \ln\left[1+T(e^{i\chi}-1)\right]^2 \\
&= \ln\left[1+T^2(e^{i2\chi}-1)+2T(1-T)(e^{i\chi}-1)\right]
\end{aligned}
$$

Different for Andreev:

$$
S_A = \ln\left[1+R_A(e^{i2\chi}-1)\right]
$$

S_A cannot be written as sum of independent terms!

Electrons in Andreev pair are strongly correlated (entangled).

Summary: approach and simple contacts

Theoretical approach: new terminal Green's functions

$$\check{G}(\chi) = e^{i\frac{\chi}{2}\check{\tau}_K}\check{G}_0 e^{-i\frac{\chi}{2}\check{\tau}_K}$$

Full counting statistics of mesoscopic two terminal contact

$$S(\chi) = \frac{t_0}{4\pi}\mathrm{Tr}\ln[1 + \frac{T_n}{4}(\{\check{G}_L(\chi), \check{G}_R\} - 2)]$$

Simple two-terminal contacts

- tunnel junction
- Levitov formula
- Andreev contact

Noise and Counting Statistics in a diffusive wire

Full Counting Statistics: Diffusive Wire

Diffusive conductor between two normal terminals
(mean free path l, length L, conductance G_N)

Transmission eigenvalue distribution

$$P(T) = \frac{l}{2L} \frac{1}{T\sqrt{1-T}}$$

Averaging the CGF: $S(\chi) = \frac{t_0}{4\pi} \int dE \mathrm{tr} \int dT P(T) \ln[4 + T (\{\check{G}_1(\chi), \check{G}_2\} - 2)]$

gives

[Lee et al. 1996, Nazarov 1999, Bagrets et al. 2003, W.B. 2003]

$$S(\chi) = \frac{G_N t_0}{8e^2} \int dE \mathrm{tr} \left[\mathrm{acosh}^2 \left(\frac{1}{2} \{\check{G}_L, \check{G}_R\} \right) \right]$$

holds for

- normal contacts
- superconducting contacts ($eV, k_B T \ll E_c$)

FCS of a Diffusive Wire

Two normal terminals

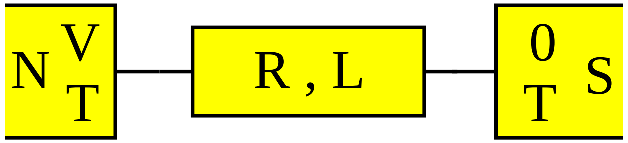
Result (at $k_B T = 0$):

$$S^{NN}(\chi) = \frac{G_N V t_0}{4e} \text{acosh}^2(2e^{i\chi} - 1)$$

- statistics is universal (conductance is the only sample parameter entering)
- Fanofactor $F = C_2/C_1 = 1/3$.
- Third cumulant $C_3/C_1 = 1/15$.

FCS of Diffusive SN-Wire

Diffusive conductor between superconductor and normal terminal
(mean free path l, length L, conductance G_N)



At $(eV, k_B T \ll E_c = \hbar D/L^2 \ll \Delta)$: coherent Andreev reflection

$$S^{SN}(\chi) = \frac{G}{2}\frac{V t_0}{4e}\text{acosh}^2(2e^{i2\chi} - 1)$$

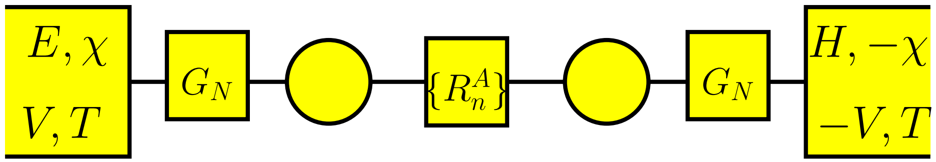
- same characteristics function as in the normal case
- statistics of doubled charge transfer ($2\chi!$)
- conductance unchanged $G_{NS} = G_N$
- Fano factor $F = 2/3$ doubled

FCS of Diffusive SN-Wire

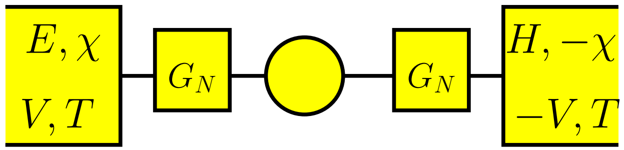
Incoherent regime ($T, V \gg E_c = D/L^2$) : W.B. and P. Samuelsson EPL 2003

only Andreev reflection at interface to superconductor

Mapping on combined electron- and hole-circuit



Interface resistance negligible $\longrightarrow$



Equivalent circuit (for normal diffusive connectors) $\longrightarrow$

Counting statistics:

$$S(\chi) = \frac{G_N}{2} \frac{V t_0}{4e} \text{acosh}^2 (2e^{i2\chi} - 1)$$

- FCS is same as in the coherent regime

FCS of Diffusive SN-Wire

- universal statistics for $E \ll E_c$ and $E \gg E_c$

Consequence of universality for $E \ll E_c$ and $E \gg E_c$:

Ratio between cumulants

$$\frac{C_n^{SN}}{C_n^{NN}} = 2^{n-1}$$

What happens in the nonuniversal regime for $E \sim E_c$?

Usadel equation

$$\nabla D(\mathrm{x})\check{G}(\mathrm{x})\nabla\check{G}(\mathrm{x}) = \left[-iE\check{\sigma}_3, \check{G}(\mathrm{x})\right]$$

Right hand side describes decoherence between electrons and holes

Normal case:

right hand side absent $\longrightarrow$Counting statistics independent of E_c

Energy-/Phase-dependent Shot Noise

Intermediate energies require "Usadel"-equation:
characteristic energy $E_c = D/L^2$ (Thouless energy)

Diffusion-like equation:
[Usadel, 70; Larkin and Ovchinikov, 68;
Eilenberger, 68; and many others]

spectral part (determines coherence):

$$D\partial_x^2\theta(E,x) = -i2E\sin(\theta(E,x))$$

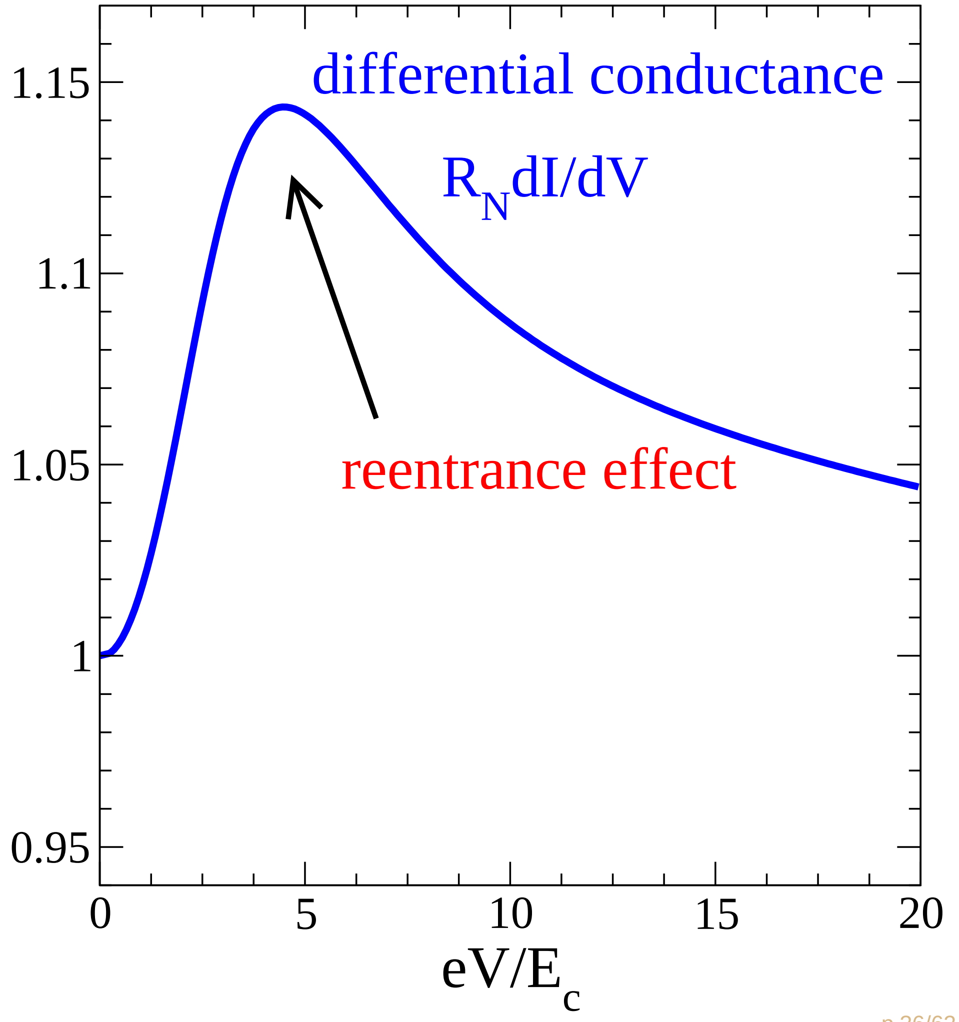
kinetic equation ("Boltzmann equation"):

$$\partial_x j = 0 \quad ; j = \sigma(E,x)\partial_x h_T$$

local energy-dependent conductivity:

$$\sigma(E,x) = \sigma_N \cosh^2(\mathrm{Re}\theta(E,x))$$

Result: [Nazarov and Stoof, PRL 96]



Energy-/Phase-dependent Shot Noise

Shot noise in the diffusive SN-wire:
Fano factor for $eV \ll E_c$ and $eV \gg E_c$: $F = 2/3$

Intermediate energy $E \sim E_c$:

Matrix 'Usadel'-equation

$$D\nabla\check{G}\nabla\check{G} = \left[-iE\check{\sigma}_3, \check{G}\right]$$

$$\check{\mathrm{I}} = -\frac{\sigma}{8}\check{G}\nabla\check{G}$$

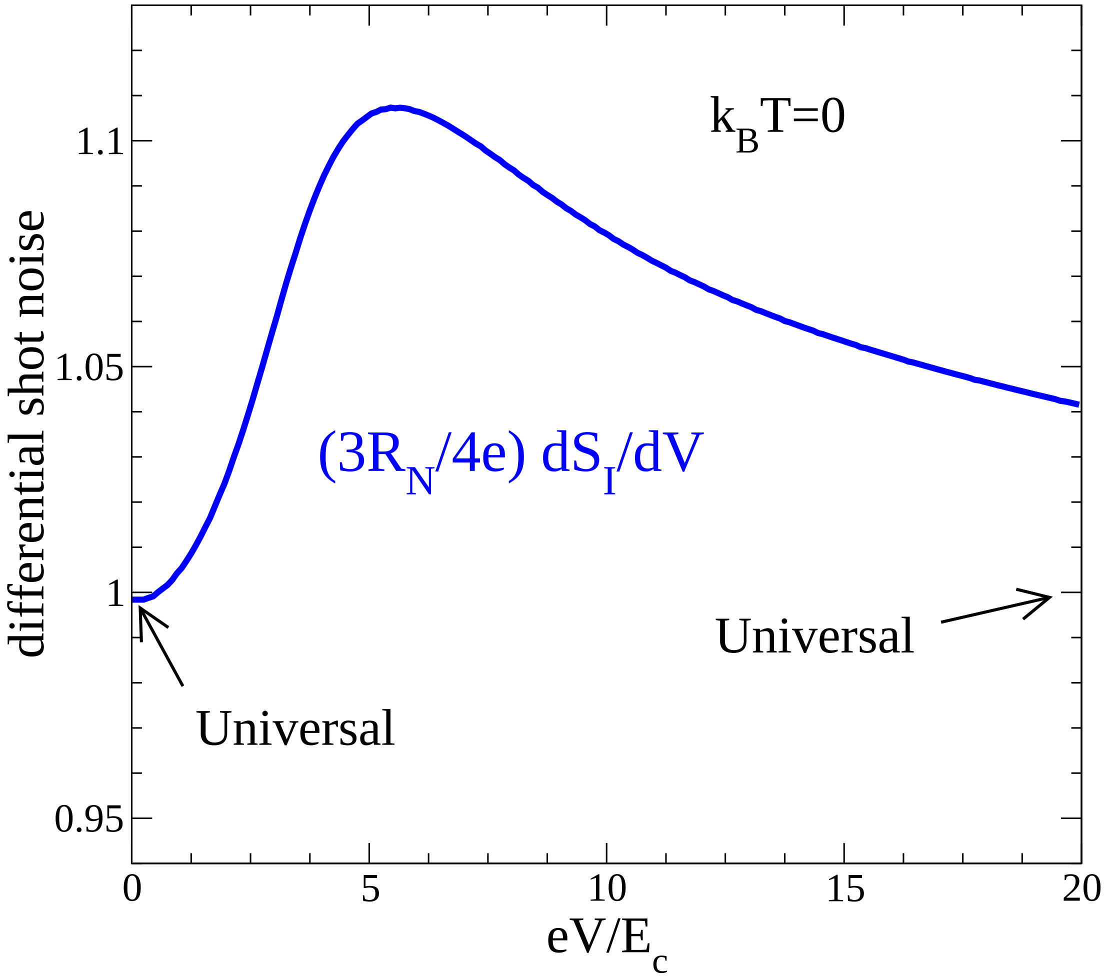
Shot noise in linear response
to counting field χ

$$\check{G}_N(\chi) = \check{G}^0_N - \chi\frac{i}{2}\left[\check{\tau}_K, \check{G}^0_N\right]$$

$$S_I = -i\frac{1}{4}\frac{\partial}{\partial\chi}\int dE\,\mathrm{Tr}(\check{\tau}_K\check{I})$$

Result:

[W.B. and Nazarov, PRL 01]



Energy-/Phase-dependent Shot Noise

Elimination of 'trivial' energy dependence:

Boltzmann-Langevin calculation with energy-dependent conductivity

$$\sigma(E,x) = \sigma_N \cosh^2(\mathrm{Re}\theta(E,x))$$

assuming independent electron- and hole-fluctuations.
Result (at $T = 0$):

$$S_I(V) = \frac{4}{3} e I(V)$$

$\longrightarrow$ we consider the effective charge

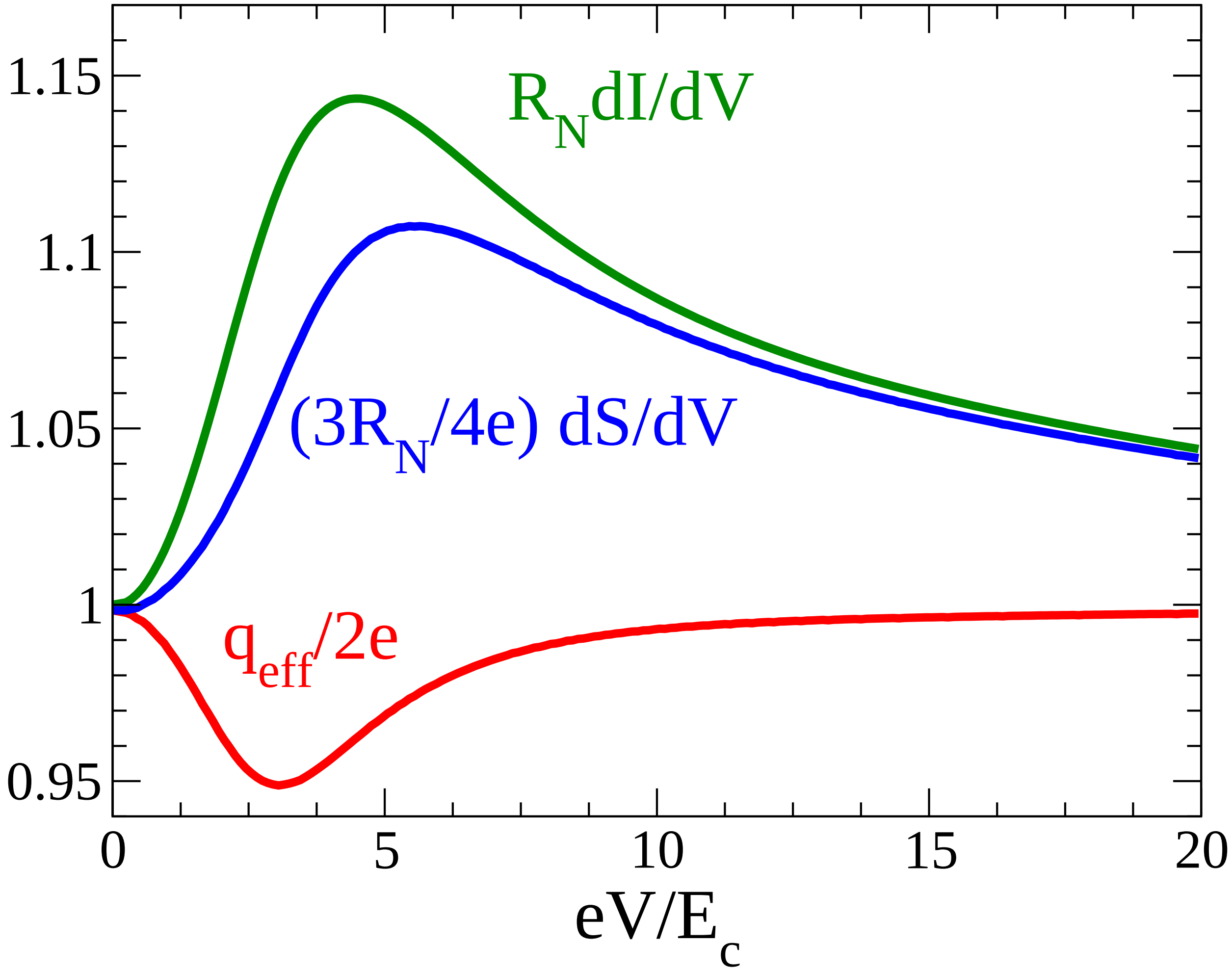
$$q_{eff}(V) = \frac{3}{2}\frac{dS_I}{dI} \left(= e \frac{C_2^{SN}}{C_2^{NN}} \frac{C_1^{NN}}{C_1^{SN}} \right)$$

Note: normalization by twice the normal state Fano factor $2/3$

Energy-/Phase-dependent Shot Noise

Effective charge:

$$q_{eff}(V) = \frac{3}{2}\frac{dS_I}{dI}$$

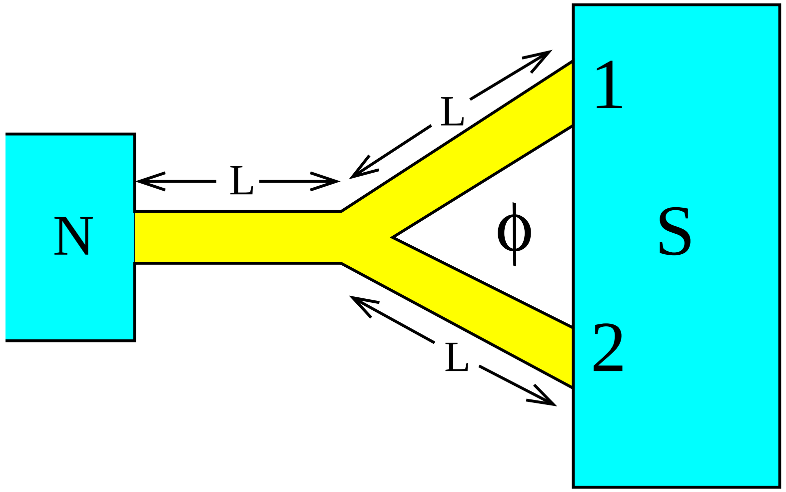


- effective charge suppressed
- origin: anti-correlated electron-hole pairs, i.e. $C_2 \sim R_A(1 - R_A)$
- enhancement of open Andreev channels for $E \lesssim E_c$

[for theory of Andreev reflection eigenvalues distribution,
see P. Samuelsson, W. Belzig and Yu. Nazarov, PRL 04]

Energy-/Phase-dependent Shot Noise

Andreev interferometer structure with a loop threaded by a magnetic flux



magnetic flux $\Phi \equiv \{$ phase difference $\delta\phi = \phi_1 - \phi_2$

characteristic energy $E_0 = \frac{D}{L^2}$

For example:

- $\phi = 0$ ($\delta\phi = 0$)
  quasi-1D-wire of length $2L$
  (non-uniform cross section)
  charact. energy $\approx E_0/4$
  size $\Delta R/R \approx 20\%$
- $\phi = \frac{1}{2}\phi_0$ ($\delta\phi = \pi$)
  destructive interference in left arm
  1D-wire of length L
  charact. energy E_0
  size $\Delta R/R \approx 20/3\%$

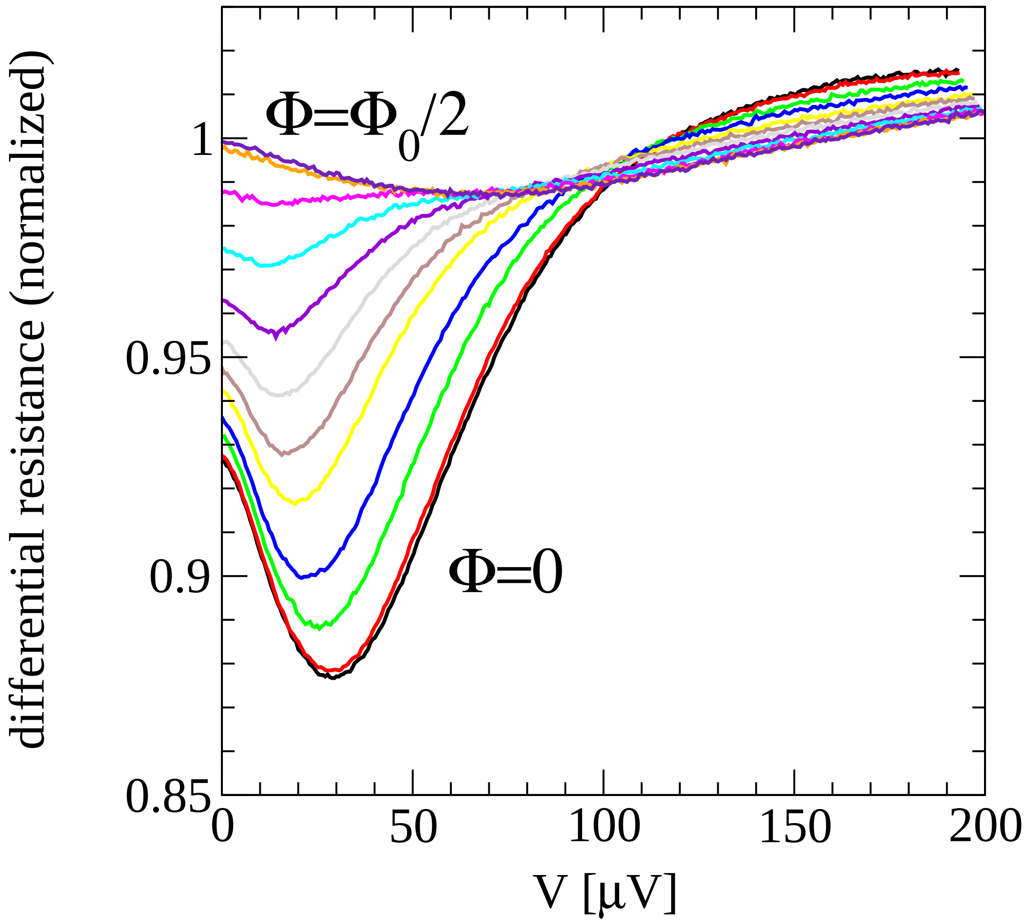
experimental realization:

[B. Reulet, A.A. Kozhevnikov, D.E. Prober, W. B., Yu.V. Nazarov, PRL 03]

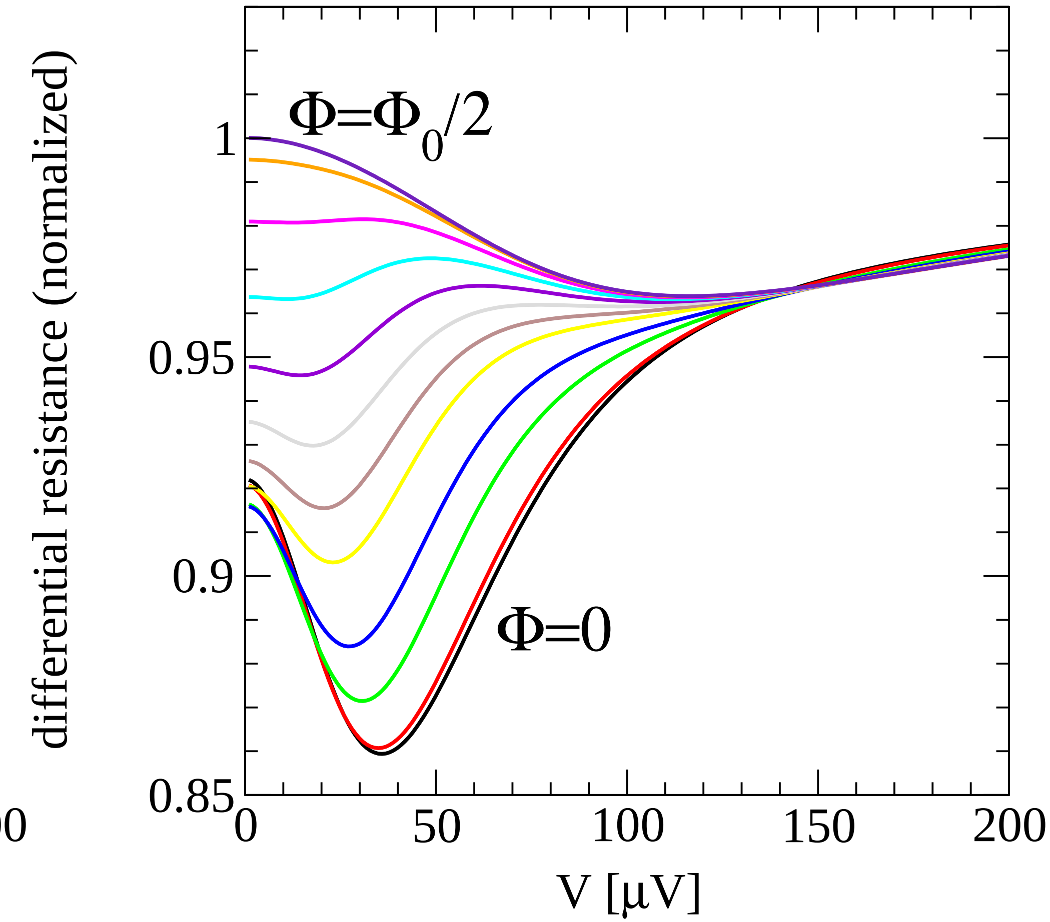
Energy-/Phase-dependent Shot Noise

Differential resistance: $T = 43$mK ; $E_c = 30\mu$eV

experiment



theory (no fit parameter)

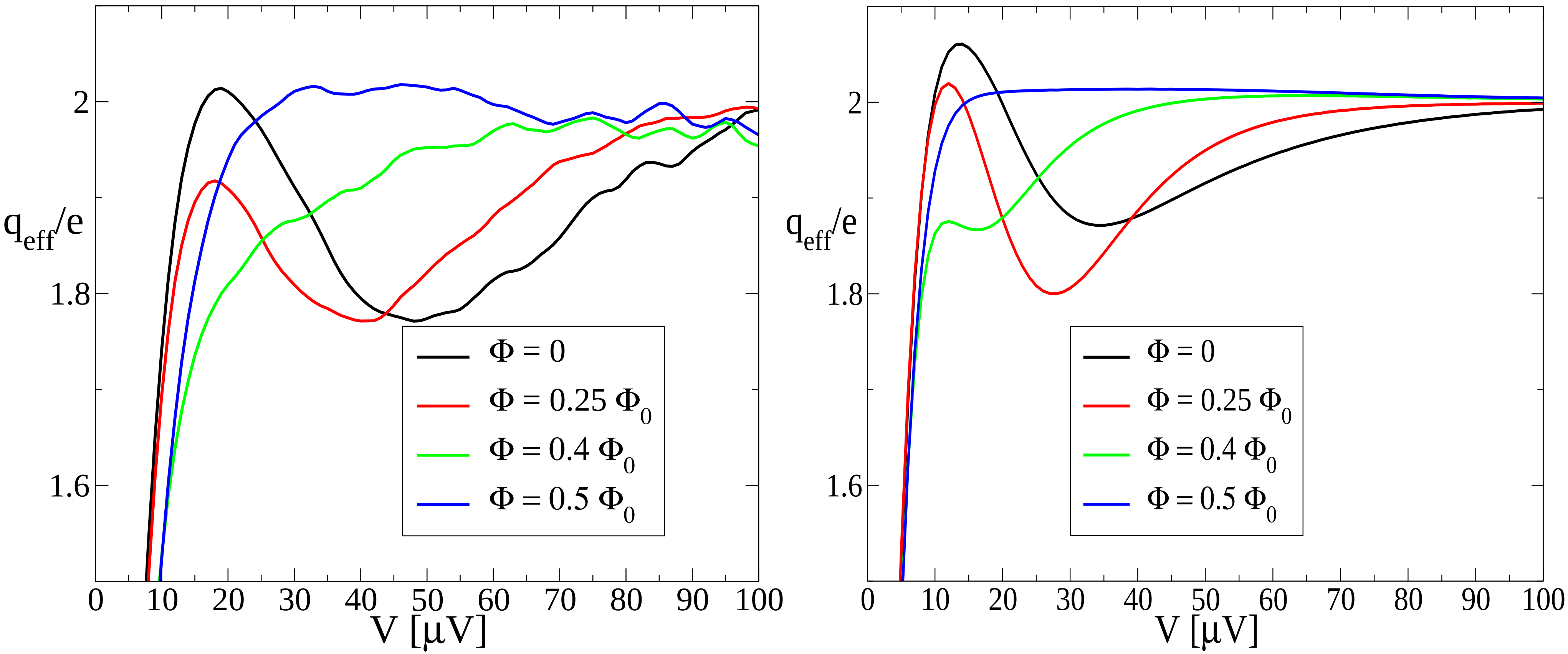


- at $\Phi = 0$: reentrance effect at $E_c \sim D/(2L)^2 = 30\mu$V.
- at $\Phi = \frac{1}{2}\Phi_0$: $1/3$ reentrance effect at $E_c \sim D/L^2 = 120\mu$V.

Energy-/Phase-dependent Shot Noise

Effective charge: $q_{\text{eff}} = (3/2)dS/dI$

experiment | theory (no fit parameter)



- at $\Phi = 0$: dip in effective charge at $E_c \sim D/(2L)^2 = 30\mu$V.
- at $\Phi = \frac{1}{2}\Phi_0$: no dip at $E_c \sim D/L^2 = 120\mu$V (both in exp. and th.!)

Summary: diffusive wire

Normal metal electrodes

- universal FCS
- no dependence on Thouless energy
- no phase effect

Superconducting contact

- universal FCS for $eV \ll E_c$ and $eV \gg E_c$
- reentrance effect for intermediate energies
- macroscopic quantum interference ($\sim \#$ of channels) no classical analog
- noise shows two-particle interference effect

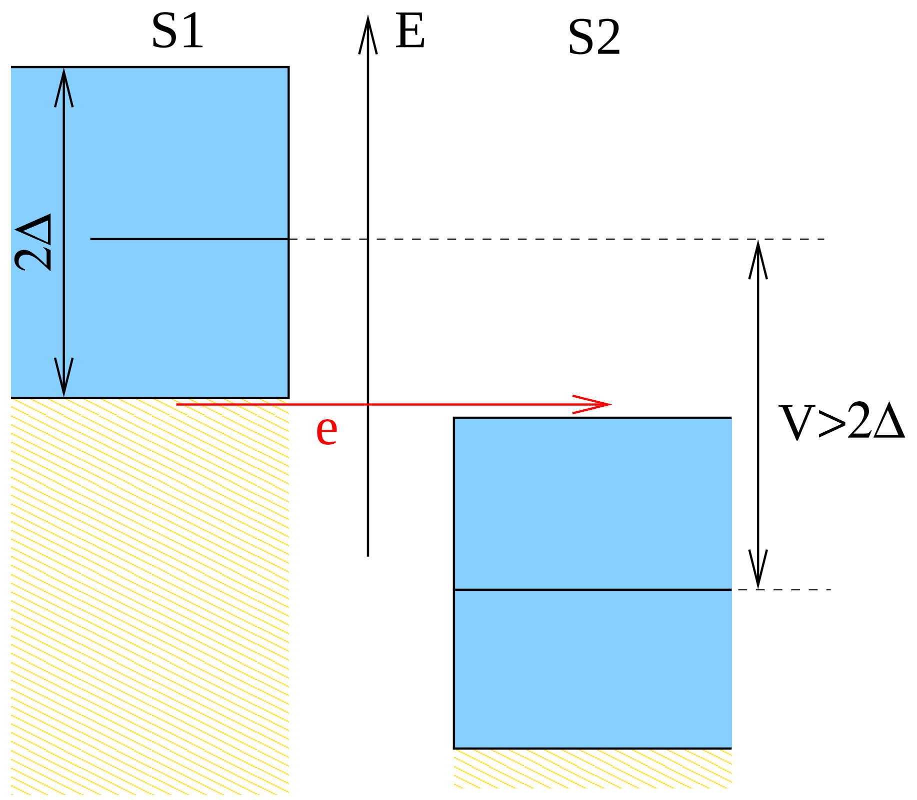
Gigantic charges in superconducting point contacts

Multiple Andreev Reflections

Superconducting junction with finite bias voltage eV
For simplicity: single channel contact with transmission eigenvalue T

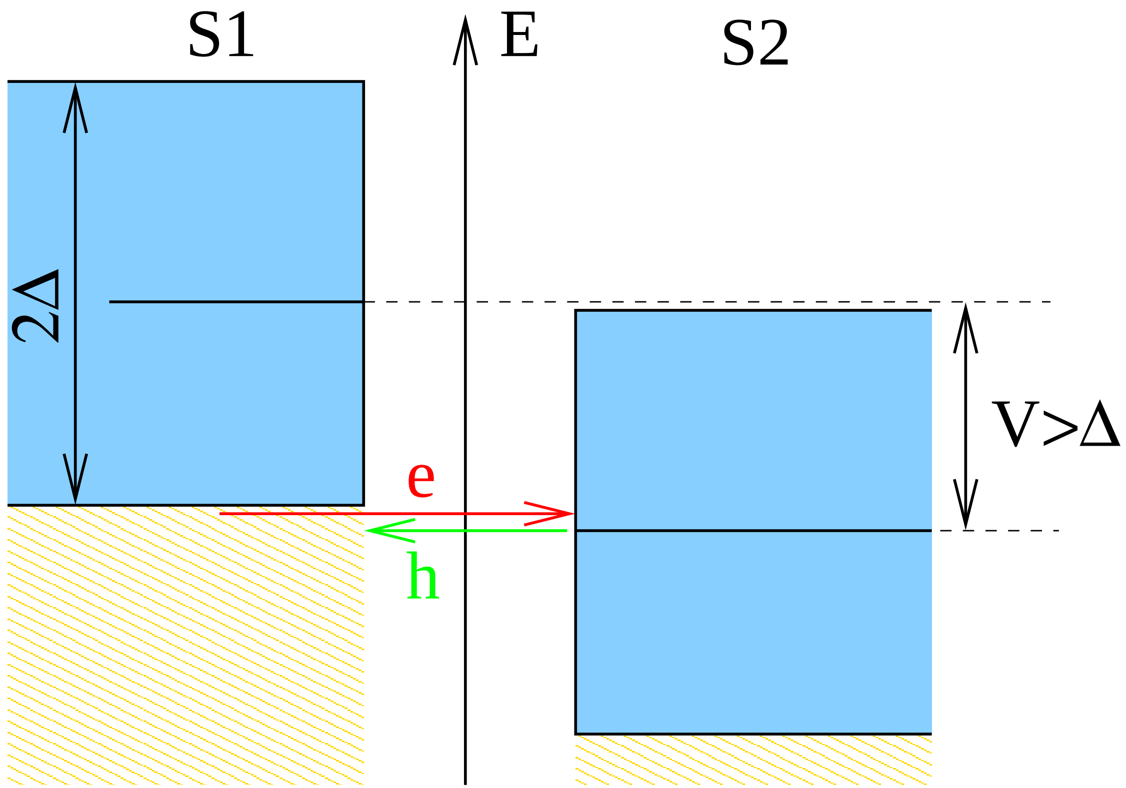
Quasiparticle tunneling:

- Total charge transfer: $1e$
- Probability: $\sim T$
- involves $\mathbf{0}$ Andreev reflections
- minimal voltage: $eV > 2\Delta$



Multiple Andreev Reflections

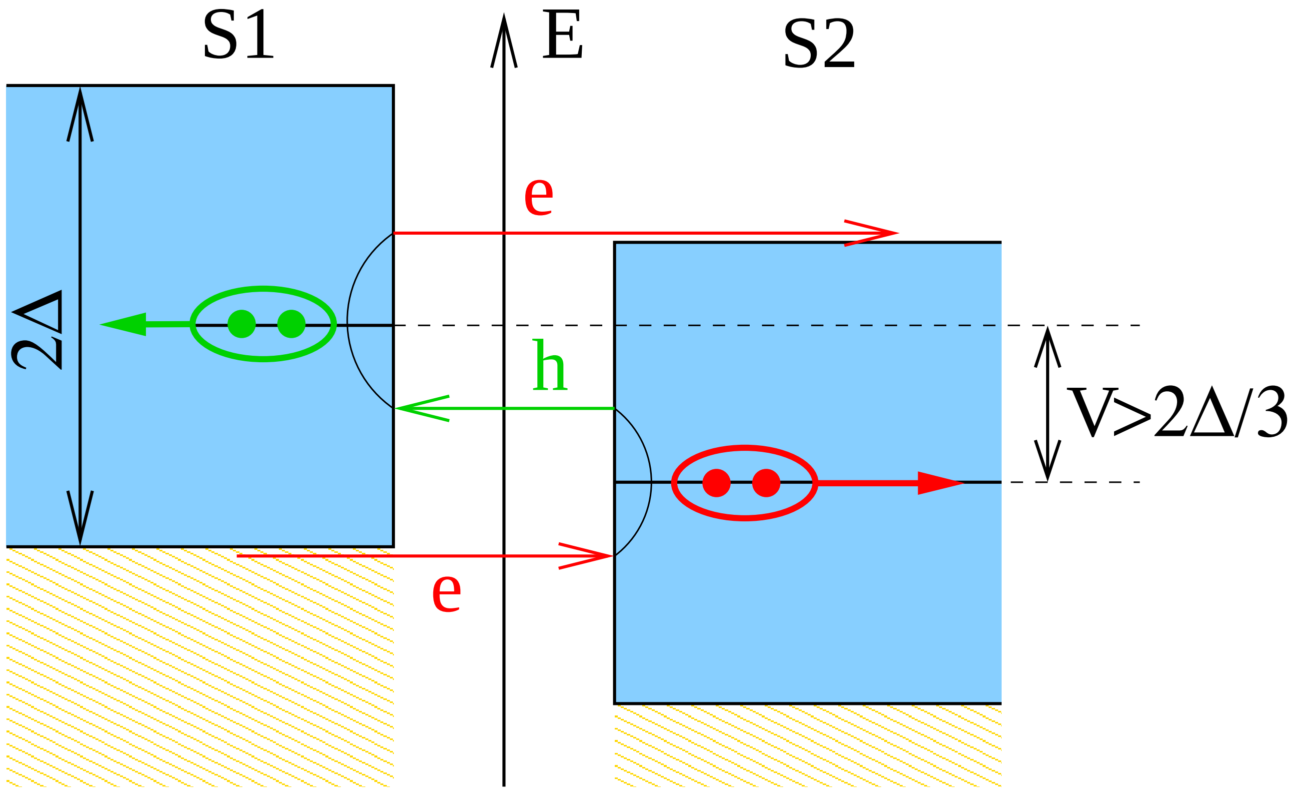
Andreev reflection (second order process):



- Total charge transfer: $2e$
- Probability: $\sim T^2$
- involves 1 Andreev reflection
- minimal voltage: $eV > 2\Delta/2$

Multiple Andreev Reflections

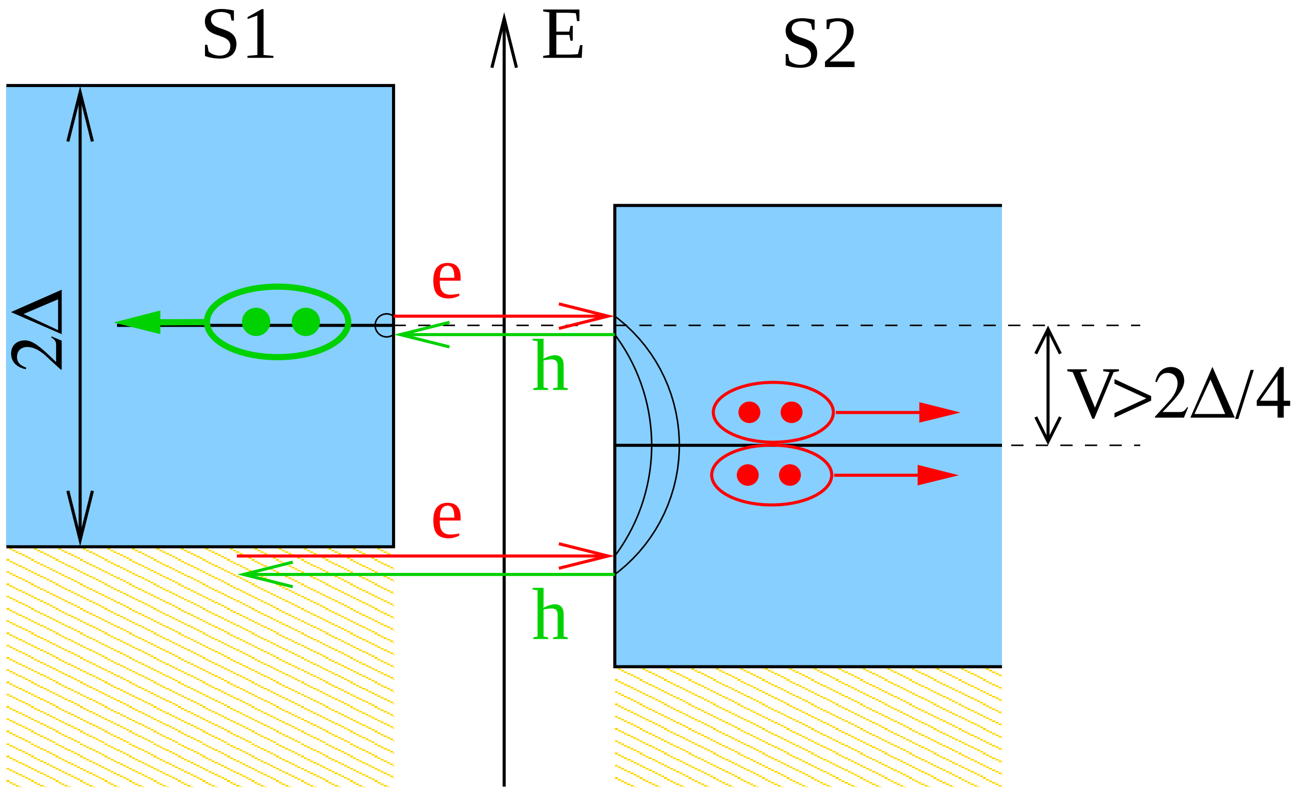
Double Andreev reflection (third order process):



- Total charge transfer: $3e$
- Probability: T^3
- involves 2 Andreev reflections
- minimal voltage: $eV > 2\Delta/3$

Multiple Andreev Reflections

Triple Andreev reflection (forth order process):

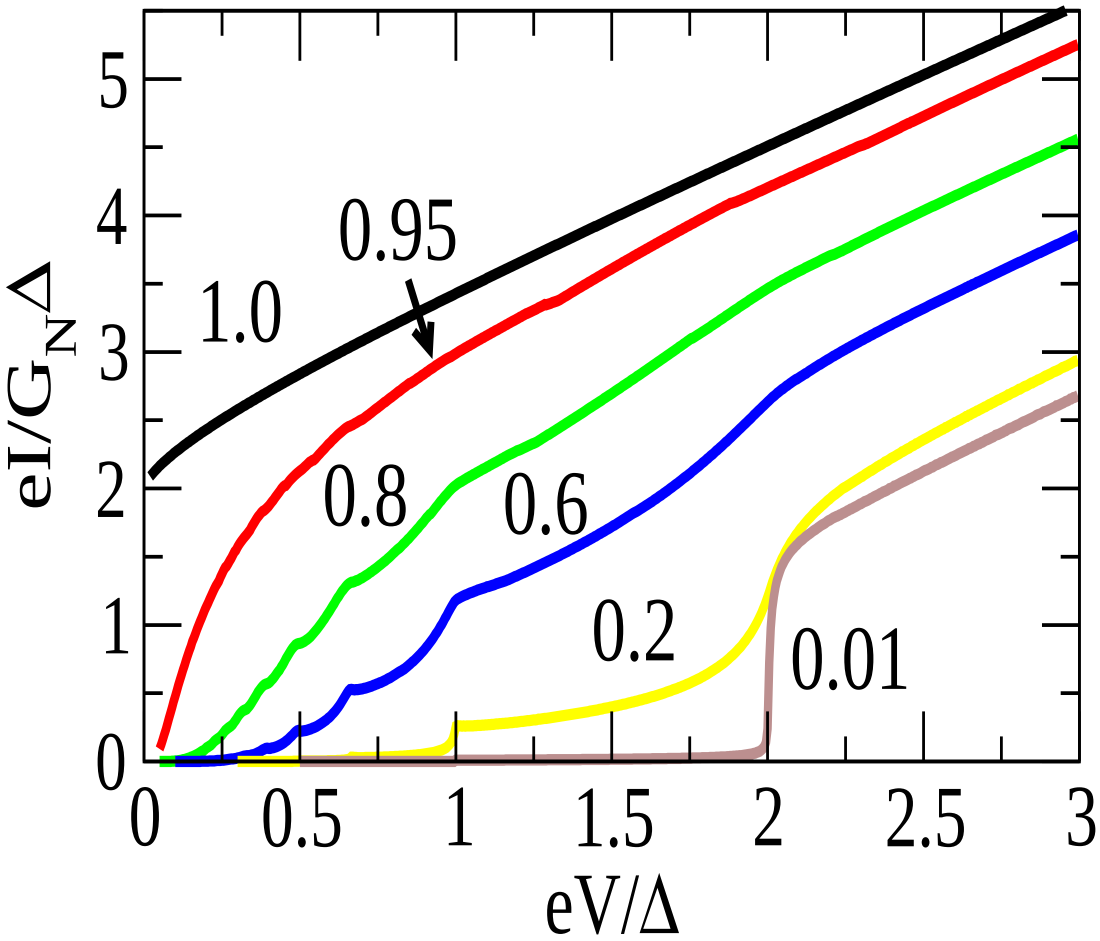


- Total charge transfer: $4e$
- Probability: $\sim T^4$
- involves 3 Andreev reflections
- minimal voltage: $eV > 2\Delta/4 = \Delta/2$

Multiple Andreev Reflections

Average current:

- strongly nonlinear current-voltage characteristic
- cusps at $eV = 2\Delta/n$
- qualitative dependence on transmission T (indicated for each curve)



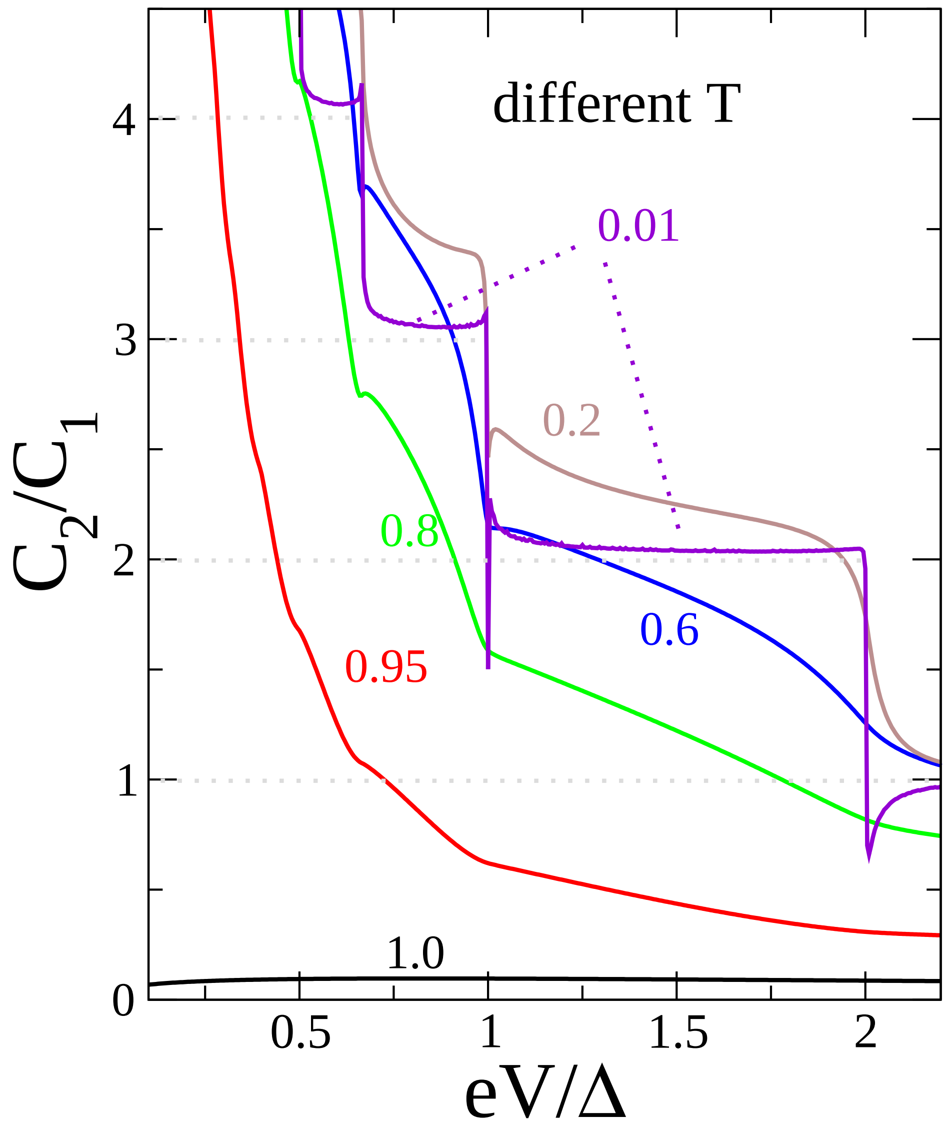
Questions:

- what are the elementary processes?
- what is their statistics?

[J.C. Cuevas et al., PRL 1996, D. Averin et al. PRL 1996]

FCS of Multiple Andreev Reflections

Example: shot noise



- $T \ll 1$ giant Fano factor: $C_2/C_1 = \text{int}(1 + 2\Delta/eV)$
- finite noise for open contact $T = 1$
- strongly enhanced noise for $eV \ll \Delta$ for any T

Questions:

- what are the elementary processes?
- what is their statistics?

[J.C. Cuevas, A. Martín-Rodero and A. Levy Yeyati, PRL 1999; Y. Naveh and D.V. Averin, PRL 1999]

FCS of Multiple Andreev Reflections

CGF of mesoscopic contact:

$$S(\chi) = \frac{t_0}{4\pi} \mathrm{Tr} \ln \left[4 + T_n \left(\left\{ \check{G}_1(\chi) \circledcirc \check{G}_2 \right\} - 2 \right) \right]$$

$$(f_1 \odot f_2)(t, t') = \int dt'' f_1(t, t'') f_2(t'', t')$$

Green's functions of two superconductors at different voltages
matrices with dimension 2×2 (Nambu $\times$ Keldysh) $\times$ $\times$ integral operator

Discretization in energy

$$\check{G}(E, E') = \sum_{nm} \check{G}_{nm}(E) \delta(E - E' + (n - m)eV$$

The $\odot$ product is reduced to usual matrix product ($\infty - dimensional$).

$2 \times 2 \times \infty =$ Nambu $\times$ Keldysh $\times$ Energy

FCS of Multiple Andreev Reflections

Structure of the matrix: (e.g. left superconductor with potential $eV/2$.

$$\check{G}_L = \left(\begin{array}{r||cc||cc||cc}
\textit{Energy} & \multicolumn{2}{c||}{n+1} & \multicolumn{2}{c||}{n} & \multicolumn{2}{c}{n-1} \\
\hline\hline
n+1 & \hat{g}_{11}^{n+1} & & & & & \\
 & & \hat{g}_{22}^{n+1} & \hat{f}_{21}^{n+1} & & & \\
\hline\hline
n & & \hat{f}_{12}^{n} & \hat{g}_{11}^{n} & & & \\
 & & & & \hat{g}_{22}^{n} & \hat{f}_{21}^{n} & \\
\hline\hline
n-1 & & & & \hat{f}_{12}^{n-1} & \hat{g}_{11}^{n-1} & \\
 & & & & & & \hat{g}_{22}^{n-1} \\
\hline\hline
\end{array}\right)$$

normal Green's functions: diagonal in energy
anomalous Green's functions: offdiagonal in energy

- matrix structure explains counting factors $e^{in\chi}$
- analytic evaluation ??
- useful formula Tr ln = ln det

FCS of Multiple Andreev Reflections

Toymodel: disregard Andreev reflection for $|E| > \Delta$

Replace Green's functions by

$$\begin{array}{lll} g^{R(A)} \to \pm 1 & f^{R(A)} \to 0 & E^2 > \Delta^2 \\ g^{R(A)} \to 0 & f^{R(A)} \to 1 & E^2 < \Delta^2 \end{array}$$

→matrix of finite dimension
Assuming $eV = 2\Delta/n$, problem is reduced to (e.g. for $n = 5$)

$$\det \left[1 - \frac{\sqrt{T}}{2} \begin{pmatrix} \hat{Q}_-(\chi) & 1 & & & \\ 1 & 0 & e^{-i\chi\hat{\tau}_3} & & \\ & e^{i\chi\hat{\tau}_3} & 0 & 1 & \\ & & 1 & 0 & e^{-i\chi\hat{\tau}_3} \\ & & & e^{i\chi\hat{\tau}_3} & \hat{Q}_+ \end{pmatrix} \right]$$

$Q_\pm$ describe quasiparticle emission (injection).

→ Determinant can be found analytically.

FCS of Multiple Andreev Reflections

Result: binomial statistics of multiple charge transfers

$$S(\chi) = \frac{eVt_0}{h} \ln \left[1 + P_n \left(e^{in\chi} - 1 \right) \right]$$

$$eV/2 = \Delta/(n-1)$$

The probabilities are

$$\begin{aligned}
P_2 &= \frac{T^2}{(T-2)^2} \\
P_3 &= \frac{T^3}{(3\,T-4)^2} \\
P_4 &= \frac{T^4}{(T^2-8\,T+8)^2} \\
P_5 &= \frac{T^5}{(5\,T^2-20\,T+16)^2} \\
P_6 &= \frac{T^6}{(T-2)^2(T^2-16\,T+16)^2} \\
P_7 &= \frac{T^7}{(7\,T^3-56\,T^2+112\,T-64)^2} \\
P_8 &= \frac{T^8}{(T^4-32\,T^3+160\,T^2-256\,T+128)^2}
\end{aligned}$$

Limits: $T = 1 \rightarrow P_n = 1$

$T \ll 1 \rightarrow P_n \sim T^n/4^{n-1}$

FCS of Multiple Andreev Reflections

Question: n^{th} order process = k quasiparticle + l Cooper pairs?

E.g. 5^{th} order

$$\det\left[1-\frac{\sqrt{T}}{2}\begin{pmatrix} \hat{Q}_-(\chi) & 1 & & & & \\ 1 & 0 & e^{-i\chi\hat{\tau}_3} & & & \\ & e^{i\chi\hat{\tau}_3} & 0 & 1 & & \\ & & 1 & 0 & e^{-i\chi\hat{\tau}_3} & \\ & & & e^{i\chi\hat{\tau}_3} & \hat{Q}_+ & \end{pmatrix}\right]$$

Possible interpretation:

- $Q_-(\chi)$ describe emission of 1 quasiparticle
- off-diagonal terms $e^{\pm i\chi\hat{\tau}_3}$ describe 1 Cooperpair

FCS of Multiple Andreev Reflections

Unitary transformations

$$\det\left[1-\frac{\sqrt{T}}{2}\begin{pmatrix} \hat{Q}_-(5\chi) & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & \hat{Q}_+ & \end{pmatrix}\right]$$

New interpretation: $Q_-(5\chi)$ describe emission of $\mathbf{5}$ quasiparticle

Question quasiparticle vs. Cooperpairs makes no sense.
Counting charges makes no distinction!

FCS of Multiple Andreev Refelections

Result of full expression: [J.C. Cuevas and W. Belzig, PRL 03]

$$S(\chi) = \frac{2t_0}{h} \int_0^{eV} dE \ln \left[1 + \sum_n P_n(E,V) \left(e^{in\chi} - 1 \right) \right]$$

Multinomial distribution of multiple charge transfers

At zero temperature analytical expressions for $P_n(E,V)$ $(n > 0)$

Cumulants:

moments of the effective charge $\langle n^k \rangle = \sum_{n=1}^{\infty} n^k P_n(E,V)$

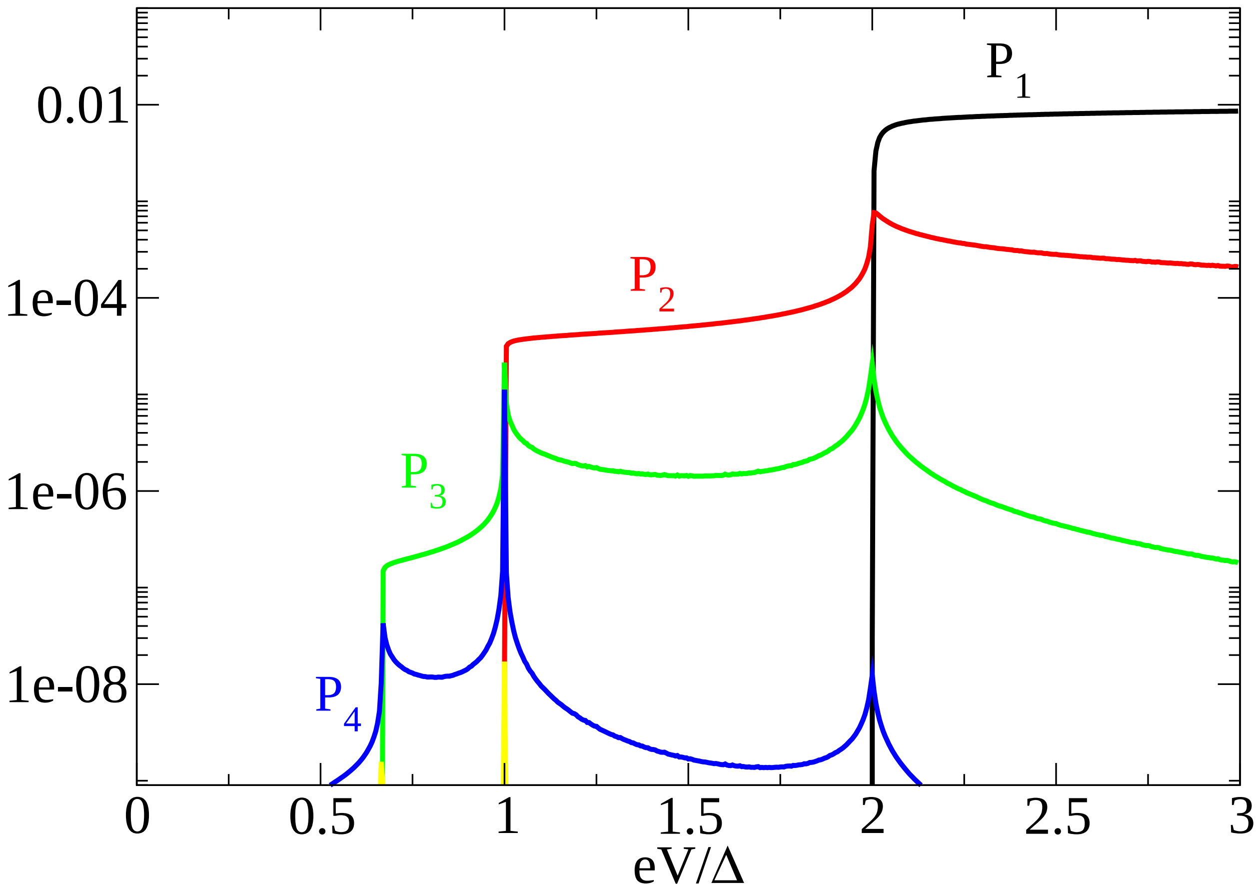
$$\begin{aligned} C_1 &= \frac{2et_0}{h} \int_0^{eV} dE \langle n \rangle \\ C_2 &= \frac{2e^2t_0}{h} \int_0^{eV} dE \langle n^2 \rangle - \langle n \rangle^2 \\ C_3 &= \frac{2e^3t_0}{h} \int_0^{eV} dE \langle n^3 \rangle - 3\langle n \rangle \langle n^2 \rangle + 2\langle n \rangle^3 \end{aligned}$$

FCS of Multiple Andreev Reflections

Tunneling regime $T = 0.01$

$$P_n = \int_0^{eV} dE P_n(E,V)/eV$$

- $\frac{\Delta}{n} < eV < \frac{\Delta}{n-1}$
  n-th order process dominates
- probability $P_n = T^n$



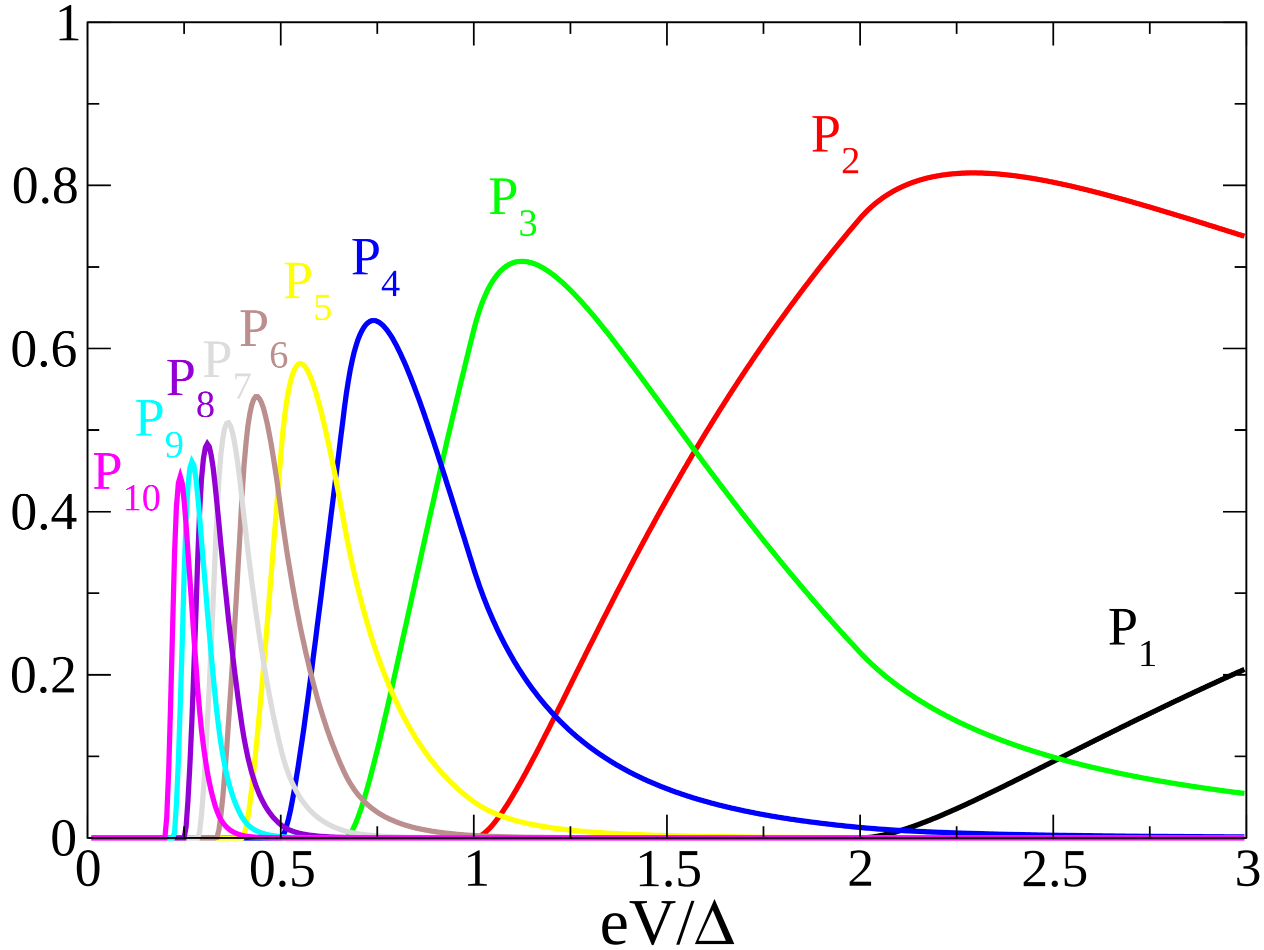
statistics is Poissonian $P_n \ll 1$

FCS of Multiple Andreev Reflections

Open contact $T = 1$

$$P_n = \int_0^{eV} dE P_n(E,V)/eV$$

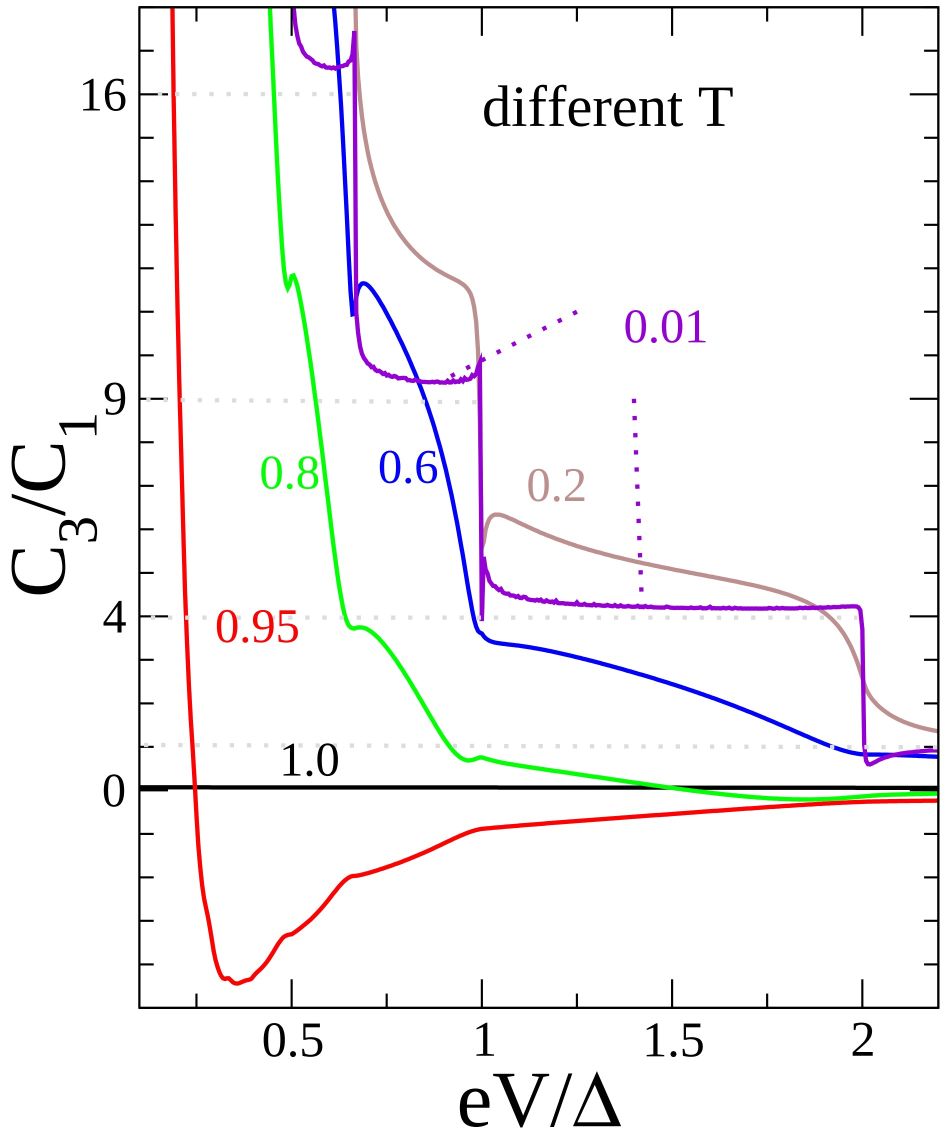
- threshold for n-th order $eV > \Delta/n$
- max of P_n for $eV = \frac{\Delta}{n-1}$
- several processes compete



full characterization of transport process!

FCS of Multiple Andreev Reflections

Example: skewness (third cumulant)



- $T \ll 1$ skewness
  $C_3/C_1 = \text{int}(1 + 2\Delta/eV)^2$
- negative skewness for ≈ 1
- strongly enhanced skewness for $eV \ll \Delta$

Summary

- Full Counting Statistics = probability of transfered charge
- Extended Keldysh-Green's function approach to FCS
- Statistics of simple two-terminal conductors
- Noise in an Andreev interferometer
  - macroscopic quantum interference
  - two-particle interference effect in noise
- Giant charges in multiple Andreev reflections
  - multinomial statistics of MAR processes

Thanks to

- Yu. V. Nazarov (Delft)
- B. Reulet, A. Kozhevnikov, D. Prober (Yale)
- J. C. Cuevas (Madrid)
- P. Samuelsson (Lund)
- C. Bruder (Basel)

Full Counting Statistics: References (incomplete)

Book (conference on quantum noise, contains many articles on full counting statistics):

- *Quantum Noise in Mesoscopic Physics*, edited by Yu. V. Nazarov (Kluwer, Dordrecht, 2003)
  Many articles are available as preprints, e.g. part of this talk [cond-mat/0210125]

Shot noise (review article):

- Ya.M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000)

Some works on Green's function approach to full counting statistics:

- Yu. V. Nazarov, Ann. Phys. (Leipzig) **8**, SI-193 (1999)
- W. Belzig and Yu. V. Nazarov, Phys. Rev. Lett. **87**, 067006 (2001).
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Keldysh-Green's functions (review article):

- J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).

Quasiclassical SN-Transport:

- W. Belzig, F.K. Wilhelm, G. Schön, C. Bruder, and A.D. Zaikin, Superlattices Microst. **25**, 1251 (1999)

Circuit theory:

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