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On distribution tail of the maximum of a random walk

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Abstract

Let S_n, $n \geqslant 1$, be the partial sums of i.i.d. random variables with negative mean value. Many papers (see, for example, [1, 2, 5, 6, 7, 9, 11]) give us different theorems on the tail behavior of the distribution of $\sup\{S_n, n \geqslant 1\}$. In this paper the final versions of these theorems (with necessary and sufficient conditions) are presented. The main attention is paid to the necessity part of these theorems. © 1997 Elsevier Science B.V.

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1. Introduction

Let $\xi_1, \xi_2, \ldots$ be an independent random variables with common distribution function $F(x)$ on $(-\infty, \infty)$; $F(+0) < 1$. Denote $\bar{F}(x) = 1 - F(x)$. Let $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$, $n \geqslant 1$, and put

$$M = \sup\{S_n, n \geqslant 0\}.$$

We assume hereafter that $\boldsymbol{E}\max(0, \xi_1)$ exists and $a = \boldsymbol{E}\xi_1 \in [-\infty, 0)$; hence S_n drifts to $-\infty$ and M is finite almost surely.

The problem is to describe the asymptotic behavior of the probability $\boldsymbol{P}\{M \geqslant x\}$ for large x. Put $\varphi(\lambda) = \boldsymbol{E}e^{\lambda\xi_1}$ and $\beta = \sup\{\lambda \geqslant 0 : \varphi(\lambda) \leqslant 1\}$. Since $\boldsymbol{P}\{\xi_1 > 0\} > 0$, $\beta < \infty$. Then only three cases can occur:

(i) $\beta = 0$, "the power tail (or subexponential) case";
(ii) $\beta > 0$ and $\varphi(\beta) < 1$, "the intermediate case";
(iii) $\beta > 0$ and $\varphi(\beta) = 1$, "the Cramér case".

It turns out that the asymptotic behavior of $\boldsymbol{P}\{M \geqslant x\}$ heavily depends on the case which takes place. To state theorems in "subexponential" and "intermediate" cases we need the following definitions.

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Definition 1. The function $f(x) > 0$ is called to be *locally power*, if, for each fixed $t, f(x+t) \sim f(x)$ as $x \to \infty$.

Definition 2. We say that the distribution G on $[0, \infty)$ with unbounded support belongs to the class $\mathscr{S}$ of *subexponential distributions* if $(G * G)([x, \infty)) \sim 2G([x, \infty))$ as $x \to \infty$. Equivalently, $\boldsymbol{P}\{\eta_1 + \eta_2 \geqslant x\} \sim 2\boldsymbol{P}\{\eta_1 \geqslant x\}$, where the independent random variables η_1 and η_2 are distributed according to G.

It is proved in Chistyakov (1964) that if $G \in \mathscr{S}$, then the function $G([x, \infty))$ is locally power. In particular, if the distribution of the random variable $\xi_1 \boldsymbol{I}\{\xi_1 \geqslant 0\}$ is in $\mathscr{S}$, then $\beta = 0$.

Definition 3. We say that the non-lattice distribution G on $[0, \infty)$ belongs to the class $\mathscr{S}(\gamma)$, $\gamma \geqslant 0$, if the function $e^{\gamma x} G([x, \infty))$ is locally power, $\int_0^t e^{\gamma t} G(dt) < \infty$, and $(G * G)([x, \infty)) \sim cG([x, \infty))$ as $x \to \infty$ for some $c \in (0, \infty)$. We say that distribution G on the lattice $\{nh, n \in \boldsymbol{Z}^+\}$ belongs to the class $\mathscr{S}(\gamma)$, $\gamma \geqslant 0$, if the previous properties hold with x taken as a multiple of lattice step h.

If $\gamma > 0$ and $G \in \mathscr{S}(\gamma)$, then it is known (see, for example, Borovkov, 1976, Section 22) that

$$\int_x^\infty G([t, \infty))\, dt \sim \frac{G([x, \infty))}{\gamma}, \quad x \to \infty, \tag{1}$$

in the case of non-lattice distribution G and

$$\int_{nh}^\infty G([t, \infty))\, dt \sim \frac{G([nh, \infty))}{1 - e^{-\gamma h}}, \quad n \to \infty, \tag{2}$$

in the case of lattice distribution G with lattice step h. Note that if the distribution of the random variable $\xi_1 \boldsymbol{I}(\xi_1 \geqslant 0\}$ belongs to $\mathscr{S}(\gamma)$ and if $\varphi(\gamma) \leqslant 1$, then $\beta = \gamma$. Clearly, $\mathscr{S} = \mathscr{S}(0)$.

Define $\tau \equiv \min\{n \geqslant 1 : S_n > 0\}$ and $\chi = S_\tau$. Since $a < 0$, τ and χ are two defective random variables. Put $p \equiv \boldsymbol{P}\{M > 0\}$. Let $\tilde{\chi}$ be a random variable with distribution

$$\boldsymbol{P}\{\tilde{\chi} \in B\} \equiv \boldsymbol{P}\{\chi \in B \mid \tau < \infty\} = p^{-1} \boldsymbol{P}\{\chi \in B\}.$$

It is known (see Feller, 1971, Ch. XII; Borovkov, 1971, Section 22) that the distribution tail of the supremum M may be calculated by the formula, $x > 0$,

$$\boldsymbol{P}\{M \geqslant x\} = (1 - p\} \sum_{k=1}^\infty p^k \boldsymbol{P}\{\tilde{\chi}_1 + \cdots + \tilde{\chi}_k \geqslant x\}, \tag{3}$$

where the random variables $\tilde{\chi}_i$ are the independent copies of $\tilde{\chi}$.

2. The subexponential case

Define the distribution $\hat{F}$ on $[0, \infty)$ by

$$\hat{F}(B) \equiv \int_B \bar{F}(t)\, dt \left(\int_{[0, \infty)} \bar{F}(t)\, dt \right)^{-1}, \quad B \subset [0, \infty).$$

Since $\boldsymbol{E}\max(0, \xi_1)$ exists, $\hat{F}$ is well defined.

Theorem 1. (A) *If $a \neq -\infty$, then these two assertions are equivalent:*

(i) $\beta = 0$ *and the distribution* $\hat{F}$ *is subexponential;*

(ii) $\boldsymbol{P}\{M \geqslant x\} \sim \frac{1}{-a}\int_x^\infty \bar{F}(t)\,\mathrm{d}t$ *as* $x \to \infty$.

(B) *If* $\beta = 0$, *the distribution* $\hat{F}$ *is subexponential, and* $a = -\infty$, *then*

$$\boldsymbol{P}\{M \geqslant x\} = \mathrm{o}\left(\int_x^\infty \bar{F}(t)\,\mathrm{d}t\right) \text{ as } x \to \infty. \tag{4}$$

(C) *If the distribution* $\hat{F}$ *has an unbounded support and* (4) *holds, then* $\beta = 0$ *and* $a = -\infty$.

Note that if the distribution of the random variable $\xi_1 \boldsymbol{I}\{\xi_1 \geqslant 0\}$ belongs to $\mathscr{S}$, then $\hat{F}$ is also subexponential. The converse assertion is not true. Indeed, the tail of the distribution F with atoms $3/4^k$ at points 2^k, $k = 1, 2, \ldots$, is not locally power and therefore, by Theorem 2 (Chistyakov, 1964), $F \notin \mathscr{S}$. Nevertheless, the corresponding distribution $\hat{F}$ is subexponential.

The implication (i) $\Rightarrow$ (ii) for so-called sub-power distributions is proved in Borovkov (1971, Section 22); in the present form it is proved in [Veraverbeke (1977)]. The implication (ii) $\Rightarrow$ (i) is proved in Embrechts and Veraverbeke (1982, Corollary 6.1) and Pakes (1975, Theorem 1) in the only case when the random variable ξ_1 is the difference of two independent random variables $\xi_1 = \eta - \zeta$, where ζ has an exponential distribution and $\eta \geqslant 0$.

Proof. (ii) $\Rightarrow$ (i): Since (see Chistyakov, 1964)

$$\liminf_{x\to\infty} \frac{\boldsymbol{P}\{\tilde{\chi}_1 + \cdots + \tilde{\chi}_k \geqslant x\}}{\boldsymbol{P}\{\tilde{\chi} \geqslant x\}} \geqslant k$$

and (see Borovkov, 1971, Section 22, Theorem 10.II)

$$\boldsymbol{P}\{\tilde{\chi} \geqslant x\} \geqslant \frac{1-p}{-ap}\int_x^\infty \bar{F}(t)\,\mathrm{d}t, \tag{5}$$

(3) implies the inequality

$$\begin{aligned}\limsup_{x\to\infty} \frac{\boldsymbol{P}\{M \geqslant x\}}{(-1/a)\int_x^\infty \bar{F}(t)\,\mathrm{d}t} &\geqslant \frac{1-p}{p}\limsup_{x\to\infty}\frac{\boldsymbol{P}\{M \geqslant x\}}{\boldsymbol{P}\{\tilde{\chi} \geqslant x\}}\\ &\geqslant \frac{(1-p)^2}{p}\sum_{k\neq 2} p^k k + (1-p)^2 p \limsup_{x\to\infty}\frac{\boldsymbol{P}\{\tilde{\chi}_1 + \tilde{\chi}_2 \geqslant x\}}{\boldsymbol{P}\{\tilde{\chi} \geqslant x\}}\\ &= 1 + (1-p)^2 p\left(\limsup_{x\to\infty}\frac{\boldsymbol{P}\{\tilde{\chi}_1 + \tilde{\chi}_2 \geqslant x\}}{\boldsymbol{P}\{\tilde{\chi} \geqslant x\}} - 2\right).\end{aligned}$$

It follows from the last inequality and the equivalence (ii) that

$$\limsup_{x\to\infty}\frac{\boldsymbol{P}\{\tilde{\chi}_1 + \tilde{\chi}_2 \geqslant x\}}{\boldsymbol{P}\{\tilde{\chi} \geqslant x\}} \leqslant 2,$$

which implies $\tilde{\chi} \in \mathcal{S}$. In particular (see Theorem 2, Chistyakov, 1964), the function $\boldsymbol{P}\{\tilde{\chi} \geqslant x\}$ is locally power. Hence, by Theorem 10.IV in Borovkov (1971, Section 22), $\hat{F}([x, \infty)) \sim \tilde{c}\boldsymbol{P}\{\tilde{\chi} \geqslant x\}$. In view of this equivalence and the property $\tilde{\chi} \in \mathcal{S}$, it follows from Lemma 2 (Pakes, 1975) that $\hat{F} \in \mathcal{S}$.

(B) It follows from the implication (i) ⇒ (ii) and the standard truncation arguments.

(C) It follows from (4) and the inequality $M \geqslant \xi_1$ that

$$\boldsymbol{P}\{\xi_1 \geqslant x\} = \mathrm{o}\left(\int_x^\infty \bar{F}(t)\,\mathrm{d}t\right) \text{ as } x \to \infty.$$

Hence, for each $\varepsilon > 0$, there exists x_0 such that

$$\bar{F}(x) \leqslant \varepsilon \int_x^\infty \bar{F}(x)\,\mathrm{d}t$$

for every $x \geqslant x_0$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \int_x^\infty \bar{F}(t)\,\mathrm{d}t \geqslant -\varepsilon,$$

which implies

$$\int_x^\infty \bar{F}(t)\,\mathrm{d}t \geqslant c\mathrm{e}^{-\varepsilon x}$$

for some $c > 0$. Since $\varepsilon > 0$ is arbitrary, $\beta = 0$.

Assume that $a \neq -\infty$. Then (5) holds. Substituting inequality (5) into (3) we conclude that

$$\boldsymbol{P}\{M \geqslant x\} \geqslant c \int_x^\infty \bar{F}(t)\,\mathrm{d}t$$

for some $c > 0$. We arrive at a contradiction. Hence $a = -\infty$. The proof of Theorem 1 is complete. □

3. The intermediate case

Theorem 2. *If $\varphi(\beta) < 1$ and if the function $\mathrm{e}^{\beta x}\bar{F}(x)$ is locally power, then the following assertions are equivalent (in the lattice case x must be taken as a multiple of the lattice step):*

(i) *the distribution of the random variable $\xi_1 \boldsymbol{I}\{\xi_1 \geqslant 0\}$ belongs to $\mathcal{S}(\beta)$;*

(ii) *as $x \to \infty$,*

$$\boldsymbol{P}\{M \geqslant x\} \sim \frac{\boldsymbol{E}\mathrm{e}^{\beta M}}{1 - \varphi(\beta)} \bar{F}(x); \tag{6}$$

(iii) $\boldsymbol{P}\{M \geqslant x\} \sim c\bar{F}(x)$ *as $x \to \infty$ for some $c \in (0, \infty)$.*

In view of (1) and (2), the relation (6) implies

$$P\{M \geqslant x\} \sim \frac{\beta E e^{\beta M}}{1-\varphi(\beta)} \int_x^\infty \bar{F}(t)\,dt \quad \text{as } x \to \infty$$

in the case of non-lattice distribution of ξ_1 and

$$P\{M \geqslant nh\} \sim \frac{(1-e^{-\beta h}) E e^{\beta M}}{1-\varphi(\beta)} \int_{nh}^\infty \bar{F}(t)\,dt \quad \text{as } n \to \infty,$$

in the case when ξ_1 has lattice distribution with lattice step h. The right-hand side term of the last relations "continuously transfers" as $\beta \downarrow 0$ to the right-hand side term of the relation (ii) in Theorem 1.

The implication (i) $\Rightarrow$ (ii) of Theorem 2 is proved in Borovkov (1976, Section 22) in the case when the function $e^{\beta x}\bar{F}(x)$ is sub-power. In the present form the implication (i) $\Rightarrow$ (ii) is formulated in Veraverbeke (1977, Theorem 2); it is proved in Bertoin and Doney (1996, Theorem 1).

Proof. (iii) $\Rightarrow$ (i): Define a random variable ξ with distribution F such that ξ and M are independent variables. It follows from the definition of the supremum that the random variables $\max\{0, M+\xi\}$ and M are equal in distribution. Hence, for $x > 0$ and $y > 0$,

$$\begin{aligned} P\{M \geqslant x\} &= \int_0^\infty P\{\xi \geqslant x - t\}\, dP\{M \geqslant t\} \\ &= \int_0^{x-y} \bar{F}(x-t)\, dP\{M < t\} - \int_{x-y}^\infty \bar{F}(x-t)\, dP\{M \geqslant t\} \\ &= \int_0^{x-y} \bar{F}(x-t)\, dP\{M < t\} + \bar{F}(y) P\{M \geqslant x - y\} \\ &\quad + \int_{-\infty}^{y} P\{M \geqslant x - t\}\, dF(t) \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{7}$$

Since the function $e^{\beta x}\bar{F}(x)$ is locally power, by condition (iii) the function $e^{\beta x} P\{M \geqslant x\}$ is also locally power and there exists a sequence $y = y(x) \to \infty$ such that

$$\frac{P\{M \geqslant x\}}{P\{M \geqslant x + h\}} \to e^{\beta h} \tag{8}$$

as $x \to \infty$ uniformly in $|h| \leqslant y(x)$. Therefore,

$$I_3 \sim P\{M \geqslant x\} \int_{-\infty}^{y} e^{\beta t}\, dF(t) \sim P\{M \geqslant x\} E e^{\beta \xi}, \quad x \to \infty. \tag{9}$$

In view of condition (iii), convergence (8), and Chebyshev inequality, we obtain

$$I_2 \equiv \bar{F}(y)\boldsymbol{P}\{M \geqslant x - y\} \sim \boldsymbol{P}\{\xi \geqslant y\} c\boldsymbol{P}\{\xi \geqslant x\}\,\mathrm{e}^{\beta y}$$
$$\leqslant \frac{\boldsymbol{E}\{\mathrm{e}^{\beta\xi}; \xi \geqslant y\}}{\mathrm{e}^{\beta y}} c\boldsymbol{P}\{\xi \geqslant x\}\,\mathrm{e}^{\beta y}$$
$$= \mathrm{o}(\boldsymbol{P}\{\xi \geqslant x\}) = \mathrm{o}(\boldsymbol{P}\{M \geqslant x\}). \tag{10}$$

Using condition (iii) and substituting (9) and (10) into (7) imply that

$$\boldsymbol{P}\{M \geqslant x\} = \frac{1 + \mathrm{o}(1)}{c} \int_0^{x-y} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\}$$
$$+ \boldsymbol{P}\{M \geqslant x\}(\boldsymbol{E}\mathrm{e}^{\beta\xi} + \mathrm{o}(1)) \tag{11}$$

as $x \to \infty$. Since

$$\int_0^{\infty} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\} = \int_0^{x-y} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\}$$
$$+ \boldsymbol{P}\{M \geqslant y\}\boldsymbol{P}\{M \geqslant x - y\}$$
$$+ \int_{-\infty}^{y} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\}.$$

almost the same as above calculations show us that

$$\int_0^{\infty} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\} = \int_0^{x-y} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\}$$
$$+ \boldsymbol{P}\{M \geqslant x\}(\boldsymbol{E}\mathrm{e}^{\beta M} + \mathrm{o}(1)) \tag{12}$$

as $x \to \infty$. It follows from relations (11) and (12) that

$$\int_0^{\infty} \boldsymbol{P}\{M \geqslant x - t\}\,\mathrm{d}\boldsymbol{P}\{M < t\} = \boldsymbol{P}\{M \geqslant x\}(\boldsymbol{E}\mathrm{e}^{\beta M} + c - c\boldsymbol{E}\mathrm{e}^{\beta M} + \mathrm{o}(1))$$
$$\equiv (\hat{c} + \mathrm{o}(1))\boldsymbol{P}\{M \geqslant x\}.$$

So, the distribution of the random variable M is in $\mathscr{S}(\beta)$. Hence, by condition (iii) and Theorem 2.7 (Embrechts and Veraverbeke, 1982), the distribution of the random variable $\xi_1 \boldsymbol{I}\{\xi_1 \geqslant 0\}$ also belongs to $\mathscr{S}(\beta)$. Theorem 2 is proved. □

4. The Cramér case

Theorem 3. *If the random variable ξ_1 has a non-lattice distribution, then the following statements are equivalent:*

(i) $\beta > 0$, $\varphi(\beta) = 1$, *and* $\varphi'(\beta) < \infty$;

(ii) $\boldsymbol{P}\{M \geqslant x\} \sim ((1 - p)/\beta\tilde{a})\mathrm{e}^{-\beta x}$ *as* $x \to \infty$, *where* $\tilde{a} \equiv \boldsymbol{E}\{\chi \mathrm{e}^{\beta\chi}; \tau < \infty\}$;

(iii) $\boldsymbol{P}\{M \geqslant x\} \sim c\mathrm{e}^{-\lambda x}$ *as* $x \to \infty$ *for some* $c, \lambda \in (0, \infty)$.

If the random variable ξ_1 *has a lattice distribution with lattice step h, then the statements* (i)–(iii) *remain equivalent if x is taken as a multiple of h and the constant* $(1-p)/\beta\tilde{a}$ *in* (ii) *is replaced by* $h(1-p)/(1-e^{-\beta h})\tilde{a}$.

The implication (i) $\Rightarrow$ (ii) is known as Cramér's estimate and may be found in Cramer (1955) and Feller (1971, Ch. XII, Section 6]. The result similar to the implication (iii) $\Rightarrow$ (i) is proved in Stadje (1995) in the only case of lattice distribution F.

Proof. (iii) $\Rightarrow$ (i): Since $\xi_1 \leqslant M$ and $\lambda > 0$, $\beta > 0$.

Assume that $\varphi(\beta) < 1$. Then, for every $\varepsilon > 0$, $\varphi(\beta+\varepsilon) = \infty$ and $\boldsymbol{E}e^{(\beta+\varepsilon)M} \geqslant \varphi(\beta+\varepsilon) = \infty$. Therefore, by the equivalence (iii), we have $\beta \geqslant \lambda$. In particular, $\boldsymbol{E}e^{\beta M} = \infty$. On the other hand, the hypothesis $\varphi(\beta) < 1$ and the inequality $e^{\beta M} \leqslant \sum_{n=0}^{\infty} e^{\beta S_n}$ imply that $\boldsymbol{E}e^{\beta M} \leqslant (1-\varphi(\beta))^{-1} < \infty$. We arrive at a contradiction. Hence, $\beta > 0$ and $\varphi(\beta) = 1$.

Assume that $\varphi'(\beta) = \infty$. Then (see Feller, 1978, Ch. XII) $\boldsymbol{P}\{M \geqslant x\} = o(e^{-\beta x})$. Since (iii), $\beta < \gamma$ and $\boldsymbol{E}e^{(\beta+\varepsilon)M} < \infty$ for some $\varepsilon > 0$. The condition $\varphi'(\beta) = \infty$ implies as well that $\varphi(\beta+\varepsilon) = \infty$. Thus, $\boldsymbol{E}e^{(\beta+\varepsilon)M} = \infty$ and we arrive at a contradiction. So, $\varphi'(\beta) < \infty$ and the proof is complete. □

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