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ON THE DISTRIBUTION DENSITY OF THE SUPREMUM OF A RANDOM WALK IN THE SUBEXPONENTIAL CASE

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Abstract: We consider a random walk $\{S_n\}$ with negative drift and heavy-tailed jumps. We study the asymptotic behavior at infinity of the distribution density of the supremum $\sup_n S_n$.

Keywords: random walk, subexponential distribution, tail asymptotics, local asymptotics

Let $\{\xi_n\}$ be a sequence of independent identically distributed random variables with common distribution $F(B) = \mathbf{P}\{\xi_1 \in B\}$ and negative mean $\mathbf{E}\xi_1 = -m < 0$. Put $S_0 = 0$, $S_n = \xi_1 + \dots + \xi_n$ and $M = \sup\{S_n, n \ge 0\}$. By the Strong Law of Large Numbers, the supremum M is finite with probability 1; we denote its distribution by $\pi(B) = \mathbf{P}\{M \in B\}$. The tail of an arbitrary distribution G we denote by $\overline{G}(x) = G((x, \infty))$.

In the present article we consider the heavy-tailed case in which $\mathbf{E}e^{\lambda\xi_1} = \infty$ for all $\lambda > 0$. In this case we study the large deviation asymptotics for the distribution of M. The most probable way how large deviations occur is well known; the main contribution is due to one big jump. As a result, the asymptotic tail behavior of the measure π is determined by the tail of F; usually it is formulated in terms of the distribution F_I introduced by the equality

$$\overline{F}_I(x) = \min\left(1, \int_x^\infty \overline{F}(y)\, dy\right), \quad x > 0.$$

In the presence of heavy tails, the class $\mathscr{S}$ of subexponential distributions is of basic importance. A distribution G on the nonnegative half-line with unbounded support is called *subexponential* if $\overline{G * G}(x) \sim 2\overline{G}(x)$ as $x \to \infty$. The next theorem is available:

Theorem 1 [1, 2]. *The following are equivalent:*
(i) *the distribution F_I is subexponential;*
(ii) $\mathbf{P}\{M > x\} \sim m^{-1}\overline{F}_I(x)$ *as* $x \to \infty$.

The problem is that in the subexponential case under consideration the integral asymptotics $\bar{\pi}(x) \sim m^{-1}\overline{F}_I(x)$ does not allow us to obtain a precise information about the local probabilities for the supremum; we can conclude only that, for any fixed t, the relation $\pi((x, x + t]) = o(\bar{\pi}(x))$ holds as $x \to \infty$. In most applications related to the local probabilities, this is not enough. It turns out that, for a local analog of Theorem 1, the key role is played by the class $\mathscr{S}^*$ consisting of all distributions G on the nonnegative half-line with unbounded support and finite mean value such that

$$\int_0^x \overline{G}(x - y)\overline{G}(y)\, dy \sim 2\overline{G}(x) \int_0^\infty \overline{G}(y)\, dy \quad \text{as } x \to \infty.$$

It is known from [3] that $G \in \mathscr{S}^*$ implies $G \in \mathscr{S}$ and $G_I \in \mathscr{S}$.

Let F_0 be a measure on $(0, \infty)$ induced by F; that is, $F_0(B) = F(B \cap (0, \infty))$.

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Theorem 2 [4]. *If $F_0/\overline{F}(0)$ is a nonlattice distribution belonging to the class $\mathscr{S}^*$ then*

$$\pi((x, x+x_0]) \sim \frac{x_0}{m}\overline{F}(x) \quad \text{as } x \to \infty \tag{1}$$

for every fixed $x_0 > 0$.

As follows from [5, Theorem 2(b)], the converse is also true: If (1) holds for every $x_0 > 0$ and F is long-tailed then $F_0/\overline{F}(0) \in \mathscr{S}^*$. Some sufficient condition for (1) is given in [6, Theorem 4.2].

In view of Theorems 1 and 2 the following question is of interest: Under what conditions has π a density in some sense (or a nonzero absolutely continuous component)? And how can we describe the asymptotic behavior of this density in the subexponential case? In particular, in what case is that density equivalent to $\overline{F}(x)/m$ as $x \to \infty$? Note that the distribution π is never absolutely continuous with respect to Lebesgue measure, since it always has an atom at the origin with weight

$$q \equiv \mathbf{P}\{M = 0\} = \mathbf{P}\{S_n \le 0 \text{ for all } n \ge 1\} > 0. \tag{2}$$

In this connection we should clarify our terminology on densities.

We say that a distribution G has *density* $g(x)$ for $x > \hat{x}$ if for any Borel set $B \subseteq (\hat{x}, \infty)$,

$$G(B) = \int_B g(y)\, dy.$$

In that case the convolution $G * G$ with necessity has density for $x > 2\hat{x}$.

We call a density g on $(\hat{x}, \infty)$ *long-tailed* (and write $g \in \mathscr{L}$) if the function $g(x)$ is bounded on $(\hat{x}, \infty)$, while $g(x) > 0$ for all sufficiently large x, and $g(x+t) \sim g(x)$ as $x \to \infty$ uniformly in $t \in [0, 1]$. In particular, if $g \in \mathscr{L}$ then $g(x) \to 0$ as $x \to \infty$.

We say that a distribution G on the nonnegative half-line with density $g(x)$ on $(\hat{x}, \infty)$ *belongs to the class $\mathscr{S}_{ac}$* if $g \in \mathscr{L}$ and

$$g^{*2}(x) \equiv 2\int_0^{\hat{x}} g(x-y) G(\,dy) + \int_{\hat{x}}^{x-\hat{x}} g(x-y) g(y)\, dy \sim 2g(x)$$

as $x \to \infty$; in that case the density g is called *subexponential*. We say that a nonnegative measure G on the nonnegative half-line with density $g(x)$ on $(\hat{x}, \infty)$ *belongs to the class $\mathscr{S}_{ac}$* if $G/G([0,\infty)) \in \mathscr{S}_{ac}$.

It follows from definitions that F_I has subexponential density if and only if $F_0/\overline{F}(0) \in \mathscr{S}^*$. In [7] the following theorem was proved in fact.

Theorem 3. *Suppose that $F_0/\overline{F}(0) \in \mathscr{S}^*$, the distribution F has density f on $(0, \infty)$, and this density is long-tailed. Then π has a density on $(0, \infty)$ asymptotically equivalent to $\overline{F}(x)/m$ as $x \to \infty$.*

In [7] Theorem 3 was formulated under a different condition. It was assumed that F has density f on $(\hat{x}, \infty)$ for some $\hat{x}$ rather than on $(0, \infty)$. Note in passing that the proof in [7] is inconsistent for $\hat{x} > 0$ since the variables $A_k(x_0)$ are incorrectly defined, in general, in the key Proposition 8 of that paper. To be more specific, the convolution F^{*n} may fail to have density for $x \le n\hat{x}$, and, for $\hat{x} > 0$, this level increases as $n \to \infty$.

Moreover, the distribution π always contains F_0 as a component with weight at least q (defined in (2)); also, each of these F_0^{*n} is a component with weight at least q as well. Hence, if the distribution F has an atom at some point $u > 0$ then the distribution π has an atom at each point ku, $k \in \mathbf{Z}^+$, with weight at least $(F\{u\})^k q$.

Let us briefly remind the history of Theorem 3. In the paper [8] the single server queue was considered with a Poisson input flow. In terms of random walk it is equivalent to the case where $\xi_1 = \sigma_1 - \tau_1$, the random variables σ_1 and τ_1 are nonnegative and independent, σ_1 is a typical service time, and τ_1 is a typical interarrival time having exponential distribution. In that case the distribution of ξ_1 has density on the whole real line; and Theorem 4.1 of [8] (in its part devoted to the subexponential case) is a particular case of Theorem 3. In Proposition 1 of [4] it was claimed that the same asymptotics as in [8] may be obtained for a general random walk. But there is no proof whatsoever and, from our point of view, the conditions stated are not correct.

In the present article, in particular, we try to understand what happens if F has density on a set $(\hat{x},\infty)$ with $\hat{x}>0$. Let us formulate and prove a statement on the absolutely continuous component of the distribution M (to be compared with the theorem of [9] dealing with an absolutely continuous component of a renewal measure).

Theorem 4. *Let the measure F_0 admit decomposition into two nonnegative measures $F_0=F_1+F_2$, where F_1 is absolutely continuous and has density $f(x)$, while F_2 is such that there is some $\lambda_2>0$ satisfying*

$$\int_{\mathbf{R}} e^{\lambda_2 x} F_2(dx) < \infty. \tag{3}$$

If $F_0/\overline{F}(0)\in\mathscr{S}^*$ and $f\in\mathscr{L}$ then the decomposition $\pi=\pi_1+\pi_2$ holds, where the measure π_1 is absolutely continuous with density asymptotically equivalent to $\overline{F}(x)/m$ as $x\to\infty$, and the tail of the measure π_2 possesses the estimate $\bar{\pi}_2(x)\le ce^{-\lambda x}$ for some $\lambda>0$.

If the distribution F has density for $x>\hat{x}$, then we deal with a particular case of Theorem 4; here the measure F_2 has bounded support. If in addition $\hat{x}=0$ then we have $F_2=0$.

In general, the measure π_1 fails to be the whole of the absolutely continuous component of π. Theorem 4 says nothing about the density of the probably nonzero absolutely continuous component of π_2. Nevertheless, it is often sufficient in various calculations to have an exponential estimate for the tail of π_2.

Proof of Theorem 4. Consider the following stopping time:

$$\eta=\inf\{n\ge 1: S_n>0\}\le\infty$$

which has a proper distribution. Let $\{\psi_n\}$ be independent identically distributed random variables with distribution

$$G(B)\equiv\mathbf{P}\{\psi_n\in B\}=\mathbf{P}\{S_\eta\in B\mid \eta<\infty\}.$$

It is well known (e.g., see [10, Chapter 12]) that the maximum M coincides in distribution with the randomly stopped sum $\psi_1+\cdots+\psi_\nu$, where the stopping time ν is independent of the sequence $\{\psi_n\}$ and is geometrically distributed with parameter $p=\mathbf{P}\{M>0\}<1$, that is, $\mathbf{P}\{\nu=k\}=(1-p)p^k$ for all $k=0,\,1,\,\ldots$; here we put the sum equal to zero on the event $\{\nu=0\}$. Equivalently,

$$\mathbf{P}\{M\in B\}=(1-p)\sum_{k=0}^{\infty}p^k G^{*k}(B). \tag{4}$$

The distribution G possesses the following representation:

$$G(B)=\int_{-\infty}^{0} F_0(B-y)H(dy),\quad B\in\mathscr{B}(0,\infty),$$

where the taboo renewal measure H is defined by

$$H(dy)=\sum_{n=0}^{\infty}\mathbf{P}\{S_j\le 0\quad\text{for all } j\le n,\ S_n\in dy\},\quad y\le 0.$$

Thus, the following decomposition of G holds:

$$G(B)=\int_{-\infty}^{0} F_1(B-y)H(dy)+\int_{-\infty}^{0} F_2(B-y)H(dy)\equiv G_1(B)+G_2(B).$$

Since the measure F_1 is absolutely continuous with density f, the measure G_1 is also absolutely continuous with density

$$g(x) = \int_{-\infty}^{0} f(x-y)H(dy), \quad x > 0.$$

By assumption $f \in \mathscr{L}$. Hence, applying Lemma 3 of [7] (with $v(x) = f(x)$) we obtain the equivalence

$$g(x) \sim (1-p)\overline{F}_1(x)/pm.$$

Since both measures F and F_1 are heavy-tailed while the measure F_2 is light-tailed (its tail is exponentially decreasing), $\overline{F}(x) \sim \overline{F}_1(x)$ as $x \to \infty$, and

$$g(x) \sim (1-p)\overline{F}(x)/pm. \tag{5}$$

In view of $F_0/\overline{F}(0) \in \mathscr{S}^*$ the density g is subexponential.

By (3), the measure G_2 satisfies Cramér's condition for all $\lambda < \lambda_2$:

$$\varphi(\lambda) \equiv \int_{\mathbf{R}} e^{\lambda x} G_2(dx) < \infty, \quad \varphi(0) = G_2(0,\infty).$$

We deduce from (4):

$$\pi = (1-p)\sum_{k=0}^{\infty} p^k \sum_{n=0}^{k} G_1^{*n} G_2^{*(k-n)} \frac{k!}{(k-n)!n!}$$

$$= (1-p)\sum_{n=0}^{\infty} p^n G_1^{*n} \frac{1}{n!} \sum_{k=n}^{\infty} p^{k-n} G_2^{*(k-n)} \frac{k!}{(k-n)!} = (1-p)\sum_{n=0}^{\infty} p^n G_1^{*n} \frac{1}{n!} H_n,$$

where the measure H_n is introduced by the equality

$$H_n \equiv \sum_{k=0}^{\infty} (pG_2)^{*k} \frac{(k+n)!}{k!}.$$

The characteristic function of H_n is equal to

$$\int_0^{\infty} e^{itx} H_n(dx) = \sum_{k=0}^{\infty} (p\psi(t))^k \frac{(k+n)!}{k!} = \frac{n!}{(1-p\psi(t))^{n+1}},$$

where $\psi(t) = \int_0^{\infty} e^{itx} G_2(dx)$; the right equality follows from the Taylor series expansion

$$\frac{n!}{(1-y)^{n+1}} = \sum_{k=0}^{\infty} y^k \frac{(k+n)!}{k!}.$$

Since $(1-p\psi(t))^{-1}$ is the characteristic function of the renewal measure

$$J \equiv \sum_{k=0}^{\infty} (pG_2)^{*k},$$

$H_n = n! J^{*(n+1)}$, and we arrive at the key equality

$$\pi = (1-p)J + (1-p)J * \sum_{n=1}^{\infty} p^n (G_1 * J)^{*n}.$$

Put

$$\pi_2 \equiv (1-p)J \quad \text{and} \quad \pi_1 \equiv (1-p)J * \sum_{n=1}^{\infty} p^n (G_1 * J)^{*n}.$$

For every $\lambda > 0$ such that $p\varphi(\lambda) < 1$, we have the following exponential estimate for the tail of J:

$$\overline{J}(x) \le e^{-\lambda x} \int_0^{\infty} e^{\lambda y} J(dy) = e^{-\lambda x} \sum_{k=0}^{\infty} p^k \varphi^k(\lambda) = \frac{e^{-\lambda x}}{1 - p\varphi(\lambda)}.$$

The distribution F is long-tailed, and so

$$J((x, x+1]) = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{6}$$

As was mentioned above, the density g of G_1 is subexponential; in particular, it is long-tailed, that is, $g(x-y)/g(x) \to 1$ as $x \to \infty$ for every fixed y. Hence, there exists an increasing level $A(x) \to \infty$ such that $g(x-y)/g(x) \to 1$ as $x \to \infty$ uniformly in $|y| \le A(x)$. Therefore,

$$\int_0^{A(x)} g(x-y)J(dy) \sim g(x)J((0,\infty)) \quad \text{as } x \to \infty.$$

In addition, (5) implies that $g(z) \le c\overline{F}(z)$ for some $c < \infty$. Thus,

$$\int_{A(x)}^{x} g(x-y)J(dy) \le c \int_{A(x)}^{x} \overline{F}(x-y)J(dy) = o\Big(\int_{A(x)}^{x} \overline{F}(x-y)\overline{F}(y)\,dy \Big) = o(\overline{F}(x))$$

as $x \to \infty$ in view of (6) and $F_0/\overline{F}(0) \in \mathscr{S}^*$. Summing up, we see as $x \to \infty$ that

$$\int_0^{x} g(x-y)J(dy) \sim J((0,\infty))g(x) = \frac{g(x)}{1 - p\varphi(0)}.$$

Therefore, the measure $G_1 * J$ has density asymptotically equivalent as $x \to \infty$ to $g(x)$ times $(1 - p\varphi(0))^{-1}$. Hence, it follows from Theorem 3 of [7] (with $\hat{x} = 0$, see the remark on Theorem 3 above) that the density of π_1 is asymptotically equivalent to $cg(x)$ as $x \to \infty$, where

$$c = (1-p)J((0,\infty)) \sum_{n=1}^{\infty} np^n G_1^{n-1}((0,\infty))J^n((0,\infty)).$$

Taking into account that $G_1((0,\infty)) = p - G_2((0,\infty)) = 1 - \varphi(0)$, we finally find the constant:

$$c = \frac{(1-p)p}{(1-p\varphi(0))^2} \sum_{n=1}^{\infty} n \Big[\frac{p(1-\varphi(0))}{1-p\varphi(0)} \Big]^{n-1} = \frac{p}{1-p}.$$

Together with (5) this completes the proof of Theorem 4.

In case (3) fails, we supplement Theorem 4 as follows:

Theorem 5. *Let the measure F_0 possess decomposition into two nonnegative measures $F_0 = F_1 + F_2$, where F_1 is absolutely continuous with density $f(x)$, and F_2 is such that there exists a distribution Q on the nonnegative half-line such that $\overline{F}_2(x) \le \overline{Q}(x)$ for all x, $Q \in \mathscr{S}^*$, and*

$$\overline{Q}(x) = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{7}$$

If $F_0/\overline{F}(0) \in \mathscr{S}^*$ and $f \in \mathscr{L}$ then the representation $\pi = \pi_1 + \pi_2$ holds, where π_1 is absolutely continuous with density asymptotically equivalent to $\overline{F}(x)/m$ as $x \to \infty$, and π_2 possesses a local estimate $\pi_2((x, x+1]) = O(\overline{Q}(x))$ as $x \to \infty$.

Proof. The measure G_2 of the previous proof enjoys the following estimate:

$$G_2((x, x+1]) = \int_{-\infty}^{0} F_2((x-y, x+1-y])H(dy) \le c_1\overline{F_2}(x)$$

for some $c_1 < \infty$. Thus, $G_2((x, x+1]) = O(Q^I((x, x+1]))$ as $x \to \infty$. Since $Q \in \mathscr{S}^*$, the distribution Q^I is locally subexponential; applying Proposition 4 of [7] we come to the upper bound

$$J((x, x+1]) = O(Q^I((x, x+1])) \quad \text{as } x \to \infty.$$

Therefore, $\pi_2((x, x+1]) \le c\overline{Q}(x)$ for some $c < \infty$. In addition, by (7) the measure J satisfies (6). Altogether, it allows us to complete the proof in the same way as in Theorem 4.

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