Knapsack Problem for Perishable Items, Index-Knapsack Heuristic, and Nearly-Optimal Revenue Management

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BCAM Workshop on Bandit Models and Applications
May 10, 2011

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Motivation

- Perishable product
  - ▷ deteriorating product with associated deadline after which it becomes worthless, if not sold
  - ▷ arises in food industry ("best before" date), fashion industry (seasonal goods), etc.
- How to select perishable products to be promoted?
  - ▷ cannot ignore time to go!
  - ▷ likely to be PSPACE-hard
- Similar problems in task management, project selection

Perishable Products

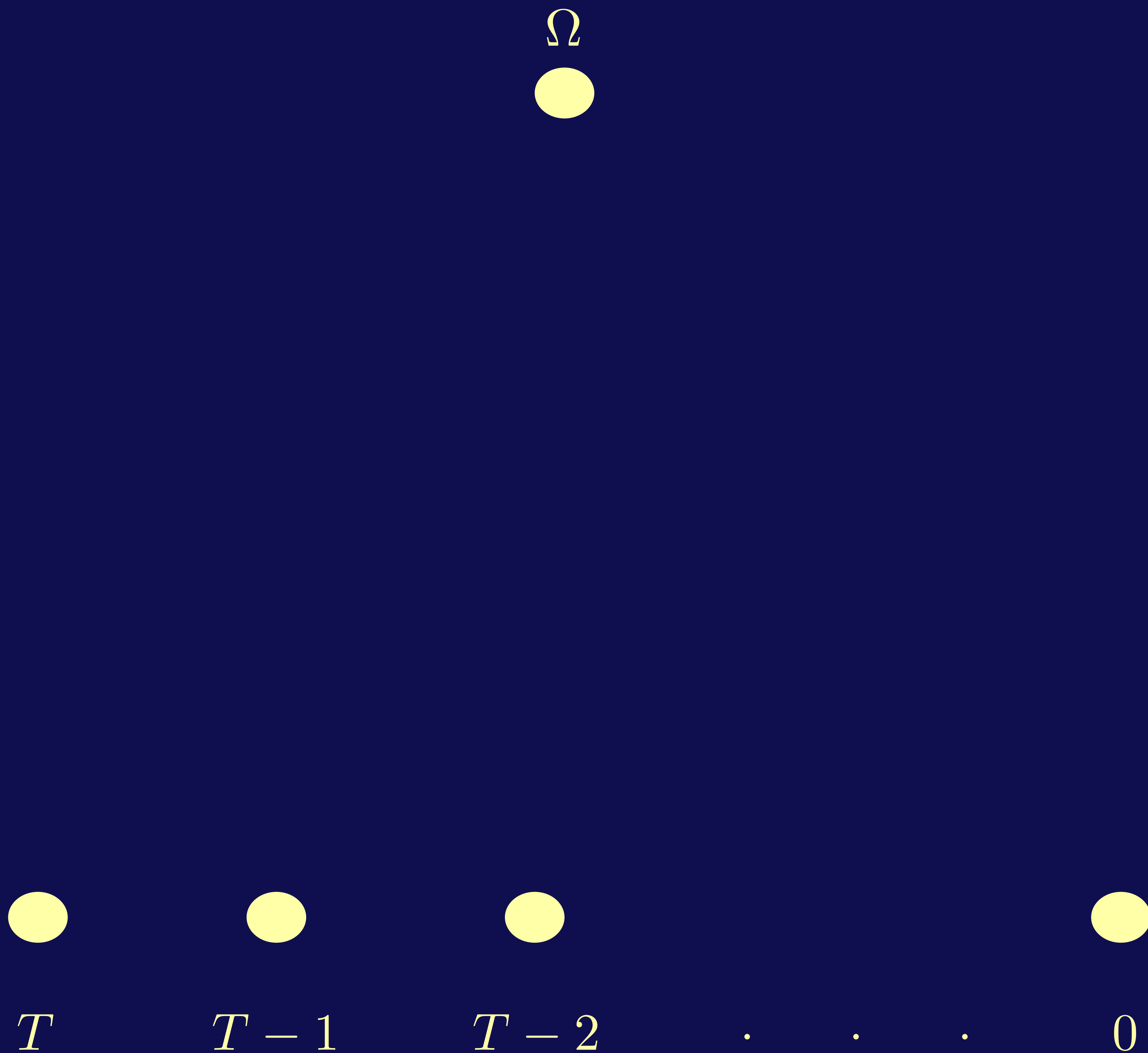
- With "increasing" demand
  - ▷ utility obtained at or after the deadline
  - ▷ e.g., transportation tickets, concert tickets, trips
  - ▷ promoted at early periods, to stimulate later demand
  - ▷ promoted at very final periods (last-minute)
- With "decreasing" demand
  - ▷ utility obtained before the deadline
  - ▷ e.g., grocery items, seasonal goods
  - ▷ promoted at final periods, to correct for wrong inventory planning, wrong pricing, or low realized demand

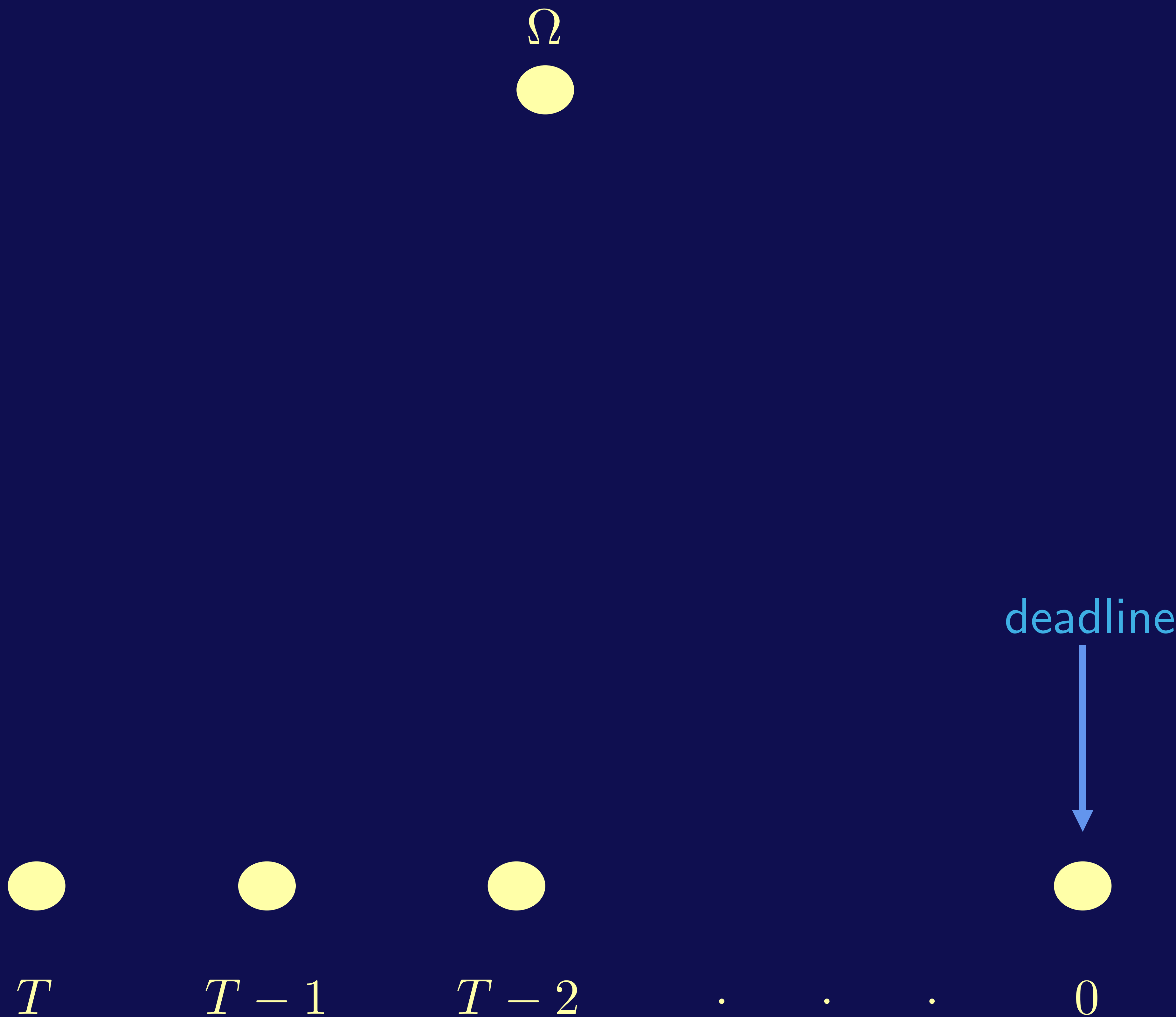
Modeling Outline

- Single-item case: Optimal Dynamic Promotion
  - ▷ Whittle index: promotion index (PI)
  - ▷ promote iff PI is larger than promotion cost
- Inventory case (omitted)
  - ▷ PI policy: calculate PI of each unit and promote iff PI is larger than promotion cost
- Network case: Knapsack Problem for Perishable Items
  - ▷ index-knapsack heuristic: calculate PI of each unit and solve a knapsack problem with PIs as item values

Characterization of a Perishable Item

- Decision moments: $s = T, T-1, \ldots, 1$
  - ▷ occupies space W, yields profit R
  - ▷ if promoted, it remains unsold with probability p
  - ▷ if not promoted, it remains unsold with probability $q > p$
  - ▷ once sold, it never resurrects
- Deadline: $s = 0$
  - ▷ yields salvage value αR, $\alpha < 1$ if not sold


Ω
T
$T-1$
$T-2$
$\cdot \quad \cdot \quad \cdot$
0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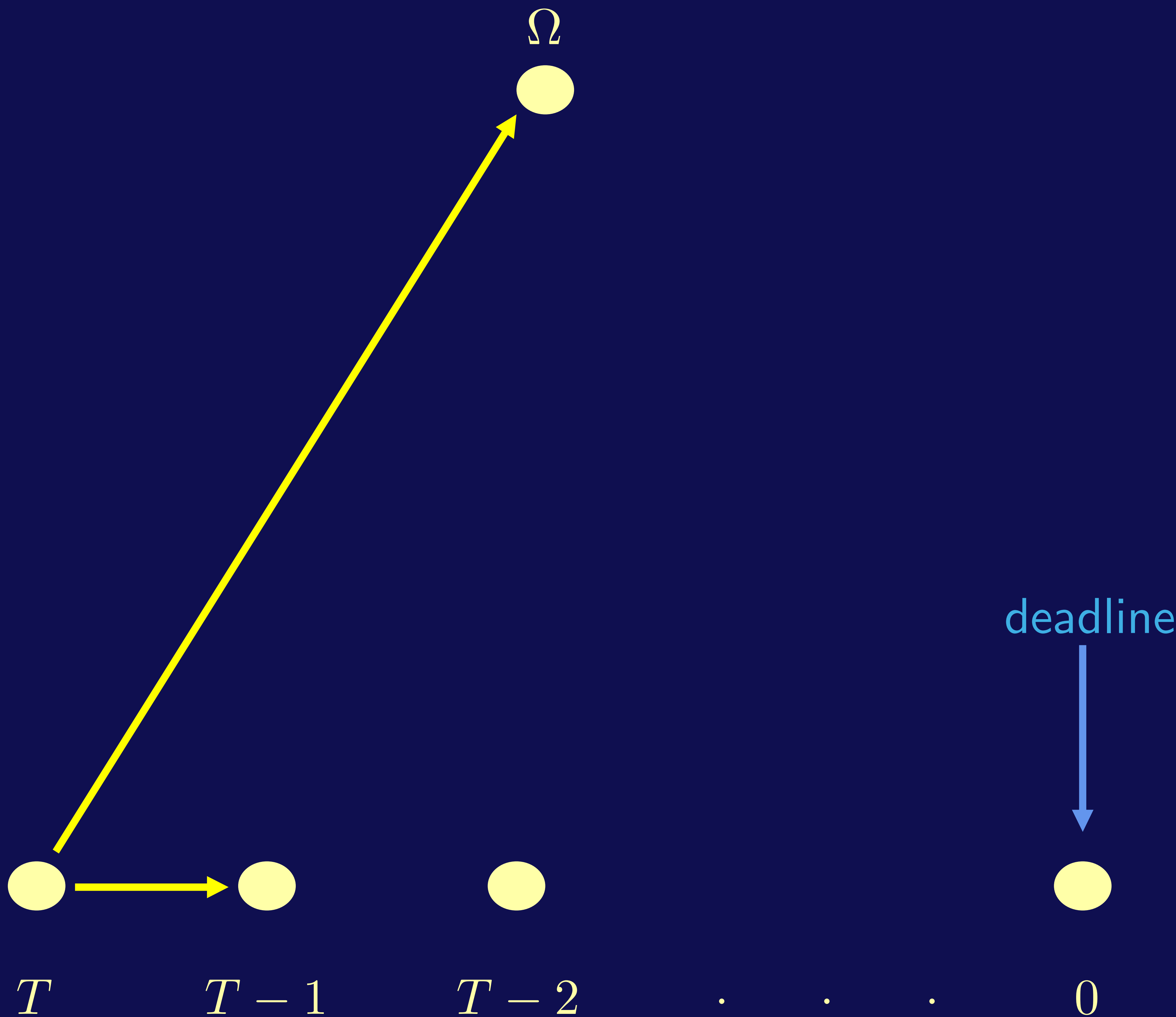
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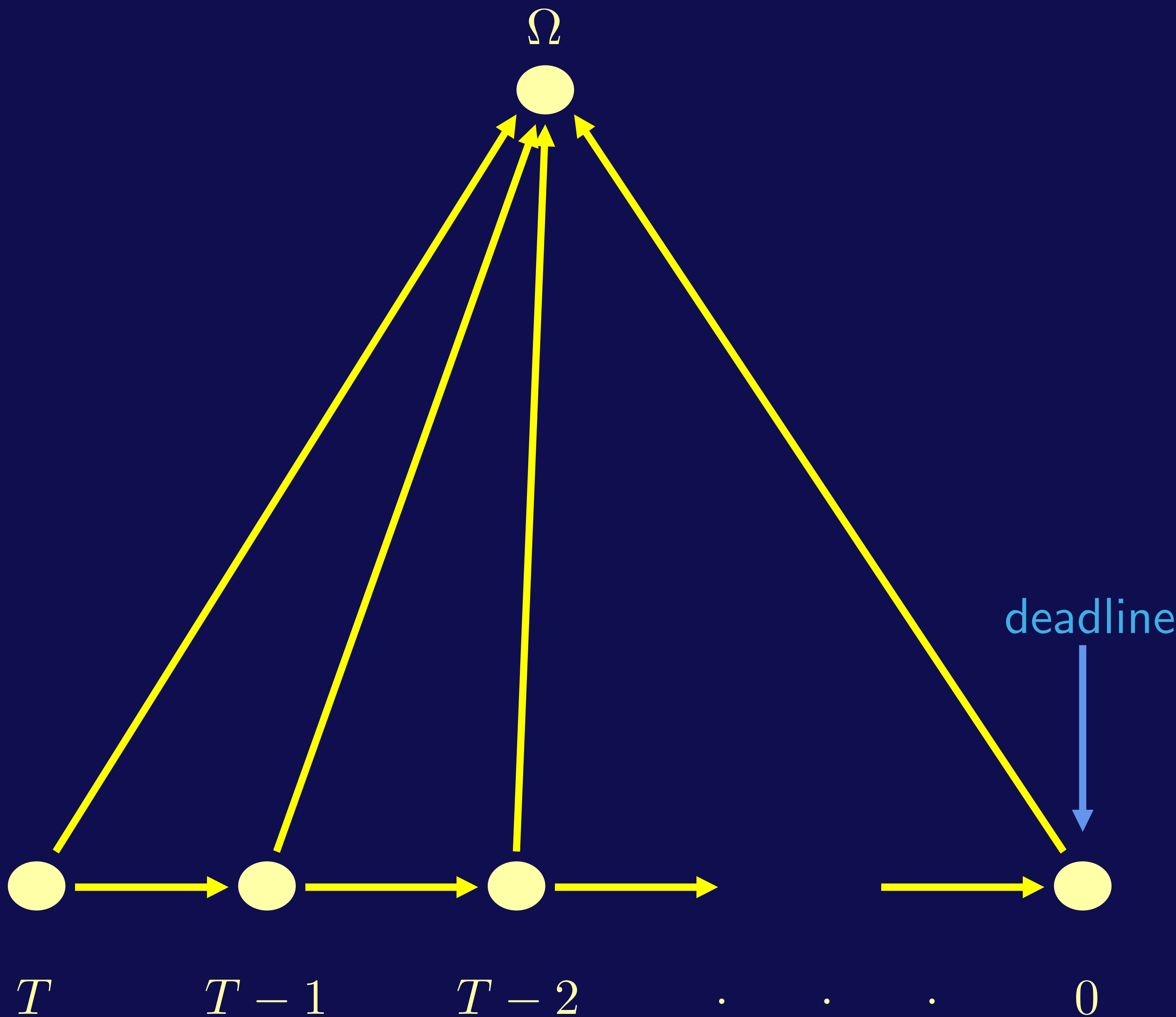

Ω
deadline
T
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$\cdot \quad \cdot \quad \cdot$
0

Perishable Item as MDP

- States:
  - ▷ $t \in \{T, T-1, \ldots, 1\}$: unsold and t periods before deadline
    - – actions to choose: promote/don't promote $\{1, 0\}$
    - – reward $R_t^1 := R(1-p)$, $R_t^0 := R(1-q)$
  - ▷ 0: unsold and perishing (exactly at deadline)
    - – no action to choose
    - – reward αR
  - ▷ Ω: sold or perished (terminal state)
    - – no action to choose
    - – no reward

The Problem

- Consider promotion cost ν per period if promoting

- Maximize the expected total β-discounted revenue:

$$\max_{\pi} \mathbb{E}_T^{\pi} \left[\sum_{s=0}^{T-1} \beta^s R_{X(s)}^{a(s)} - \nu \sum_{s=0}^{T-1} \beta^s W_{X(s)}^{a(s)} \right]$$

or simply $$\max_{\pi} \mathbb{R}_T^{\pi} - \nu \mathbb{W}_T^{\pi}$$

- $R_{X(s)}^{a(s)}$ is the reward at time s

- $W_{X(s)}^{a(s)}$ is the "promotion work" at time s $(0/W)$

Intuitive Solution

- Expected properties of optimal solution:
  - ▷ if optimally promoted for ν,
    then optimally promoted for $\nu' < \nu$
  - ▷ if optimally promoted at t,
    then optimally promoted at $t - 1$

- Aim: To each state t assign promotion index value ν_t so that it is optimal to promote at state t iff $\nu_t > \nu$

- We expect $\nu_t \leq \nu_{t-1}$ (increasing as deadline approaches)

Promotion Index (PI)

- Stationary policy $\pi \equiv$ promotion set $\mathcal{S} \subseteq \mathcal{T}$
- PI ν_t for state t must satisfy: if $\nu = \nu_t$,
  both promoting and not promoting are optimal
- So, there is a promotion set $\mathcal{S}_t$ for state t such that

$$\mathbb{R}_t^{\mathcal{S}_t \cup \{t\}} - \nu_t \mathbb{W}_t^{\mathcal{S}_t \cup \{t\}} = \mathbb{R}_t^{\mathcal{S}_t \setminus \{t\}} - \nu_t \mathbb{W}_t^{\mathcal{S}_t \setminus \{t\}}$$

- Therefore, if denominator is nonzero,

$$\nu_t = \frac{\mathbb{R}_t^{\mathcal{S}_t \cup \{t\}} - \mathbb{R}_t^{\mathcal{S}_t \setminus \{t\}}}{\mathbb{W}_t^{\mathcal{S}_t \cup \{t\}} - \mathbb{W}_t^{\mathcal{S}_t \setminus \{t\}}} \quad \text{for some } \mathcal{S}_t$$

Interpretation of PI

- Marginal rate of substitution for promoting
- Marginal productivity rate of promoting with respect to not promoting
- Expected marginal reward divided by marginal work
- History of indices:
  - ▷ $c\mu$-rule (1950s)
  - ▷ Gittins' index (1970s)
  - ▷ Whittle's index (1988)
  - ▷ MPI: Niño-Mora (2000s)

PI for Perishable Item

- Under a regularity condition

$$(1-q)-\alpha(1-\beta q)\geq 0$$

we have

$$\mathcal{S}_t=\{t,t-1,\ldots,1\}$$

- Closed-form formula:

$$\nu_t=\frac{R}{W}\left\{[(1-p)-\alpha(1-\beta p)]-\frac{[(1-q)-\alpha(1-\beta q)](1-\beta p)}{(1-\beta q)+(\beta q-\beta p)(\beta p)^{t-1}}\right\}$$

PI Properties

- Nonnegative and proportional to R/W
- Increasing in q
- Nondecreasing as deadline approaches: $\nu_t \leq \nu_{t-1}$
- Extends to undiscounted case $(\beta = 1)$
- Extends to non-perishable items

$$\nu_t \to \frac{R}{W}\frac{(1-\beta)(q-p)}{1-\beta q} \text{ as } t \to \infty$$

Knapsack Problem for Perishable Items

- Consider I perishable items
- Item i occupies space W_i
- Let C be the promotion space (knapsack)
- A dynamic and stochastic combinatorial problem
- Aim: Fill in the knapsack so that the expected aggregate total β-discounted revenue is maximized

KPPI $\rightarrow$ KP Reduction

- KPPI reduces to Knapsack Problem
  when $T_i = q_i = 1$, $p_i = 0$
- (KP) is NP-hard $\implies$ KPPI is at least NP-hard
- In fact, KPPI seems to be PSPACE-hard, because it is restless

Dynamic Programming Formulation

$$D_T(\boldsymbol{z}_T) = \sum_{i \in \mathcal{I}_T^0} c_i z_{(T,i)}$$

$$D_s(\boldsymbol{z}_s) = \sum_{i \in \mathcal{I}_s^0} c_i z_{(s,i)} + \min_{\substack{\boldsymbol{y}_s \le \boldsymbol{z}_s^+ \\ \sum_{i \in \mathcal{I}_s^+} W_i y_{(s,i)} \le C}} \left\{ \sum_{\boldsymbol{m}_s \le \boldsymbol{z}_s^+} \mathbb{P}^{\boldsymbol{y}_s}[\boldsymbol{m}_s] D_{s+1}(\boldsymbol{z}_s^+ - \boldsymbol{m}_s) \right\}$$

- Solving a system of an exponential number of equations for an exponential number of vectors $\boldsymbol{z}_s$ at every stage
  - ▷ tractability problem: curse of dimensionality
  - ▷ no interpretation

Index-Knapsack Heuristic for KPPI

- Index-knapsack (IK) heuristic:
- 1. Compute promotion index
- 2. Solve 0-1 Knapsack Problem for items $i \in \mathcal{I}$:

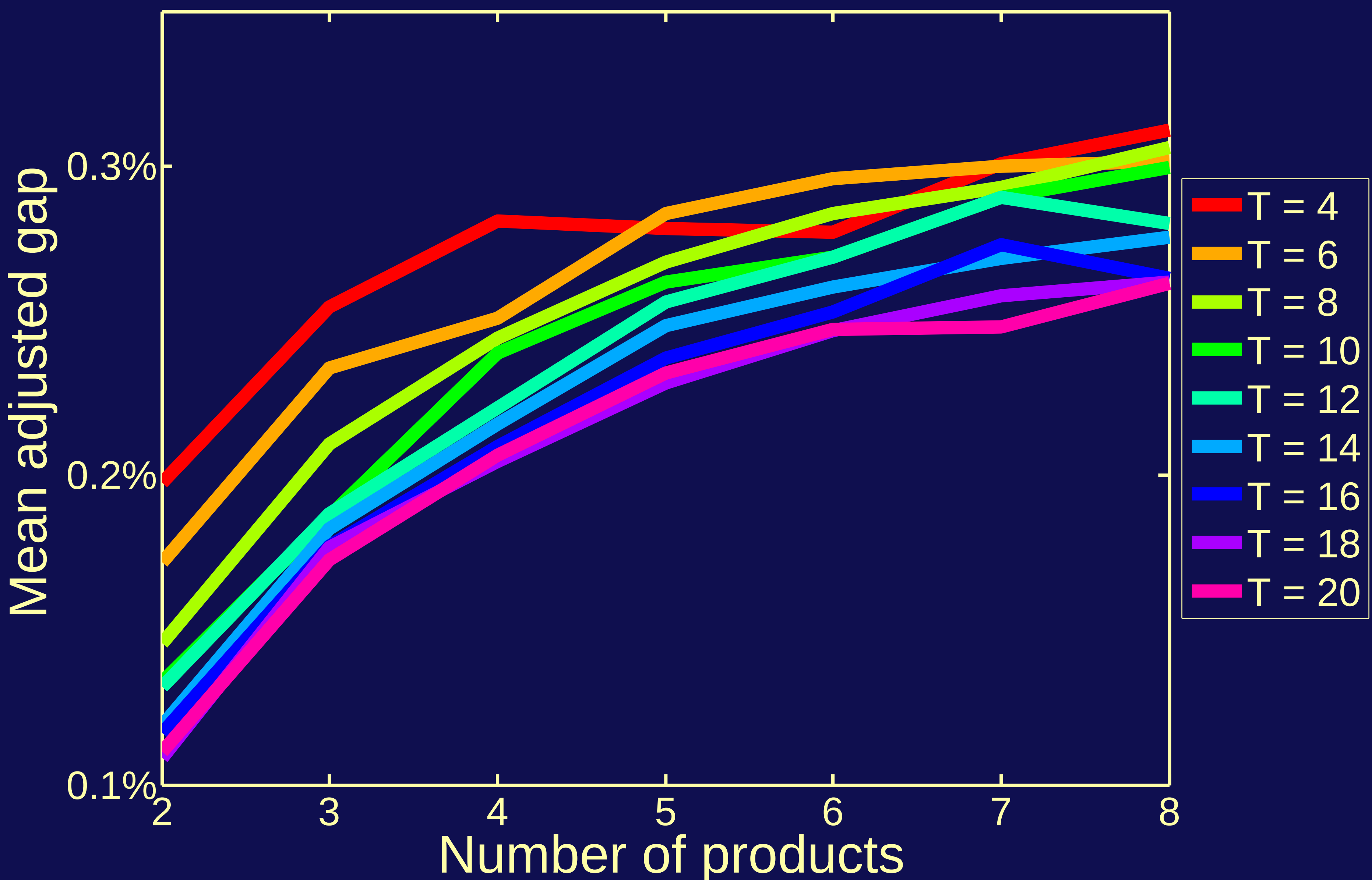
$$\begin{aligned} \max_{\boldsymbol{x}} \sum_i \nu_{i,T} x_i & \\ \text{subject to } \sum_i w_i x_i \leq C & \qquad \text{(KP)} \\ x_i \in \{0,1\} \text{ for all } i & \end{aligned}$$

- 3. Promote item i iff $x_i = 1$

Experimental Study

- Randomly generated instances, $R_i, W_i \in [10, 50]$
- Let $T = \max\{T_i\}$ be the time horizon
- Poisson demand with rate λ_i
- Inventory planning: $\frac{1}{2}\lambda_i T_i \leq J_i \leq \frac{3}{2}\lambda_i T_i$
- Knapsack volume W less than 30% of total volume
- Experiment: I, T varying (10000 instances)

Performance of IK Heuristic



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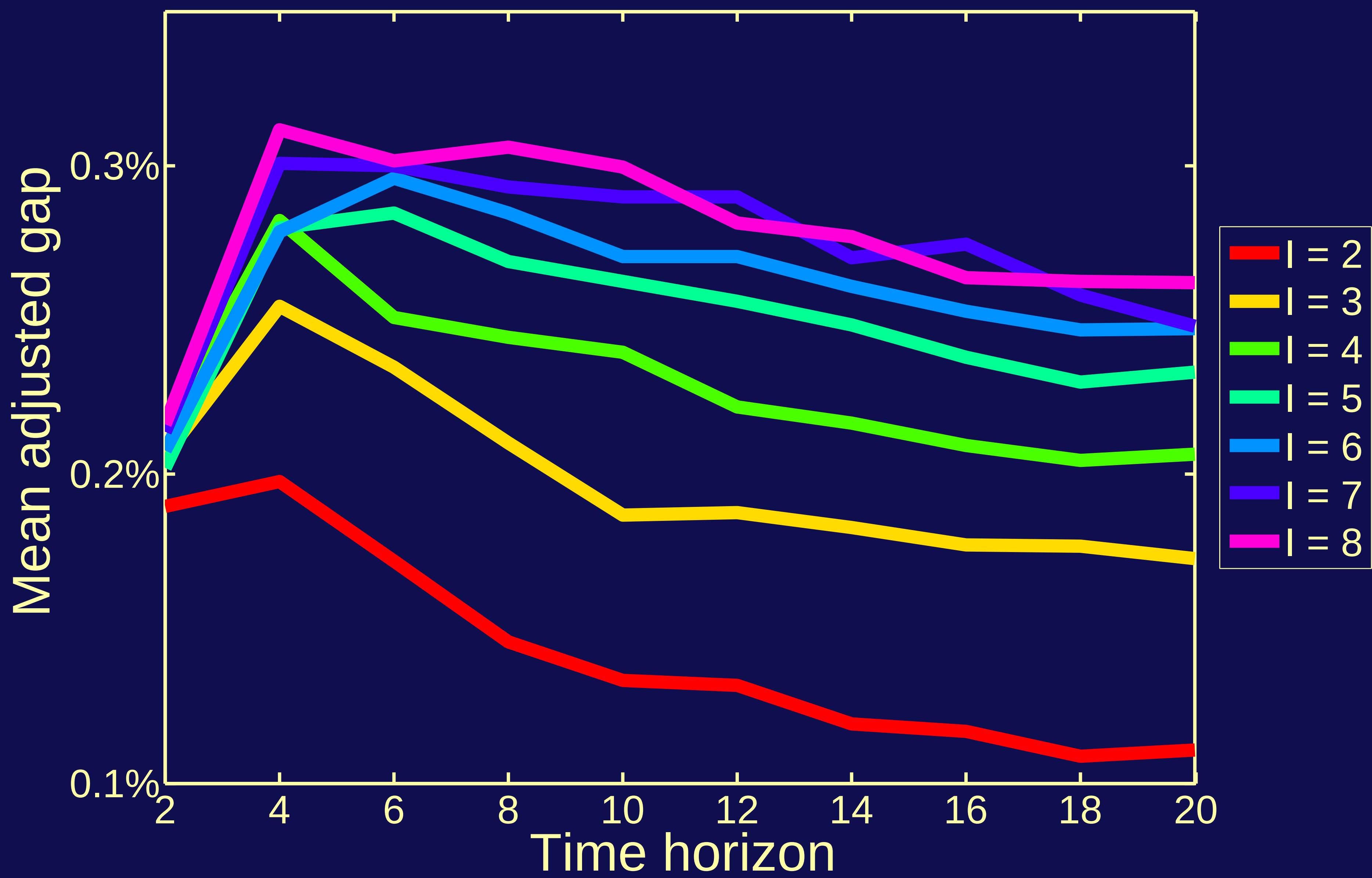


Relative Suboptimality Gap

$$\mathrm{rsg}(\pi) = \frac{C^{\pi} - C^{\min}}{C^{\min}}$$

- Takes values between 0 (achievable) and ∞ (?)
- For what values of $\mathrm{rsg}(\pi)$ is π a "good" policy?
- Generally accepted: below 5%
- Is it a good measure for bounded-from-above problems?
- What if $\mathrm{rsg}(\max) = 10\%$? What if $C^{\min} \approx 0$?

Adjusted Relative Suboptimality Gap

$$\text{arsg}(\pi) = \frac{C^{\pi} - C^{\min}}{C^{\max} - C^{\min}}$$

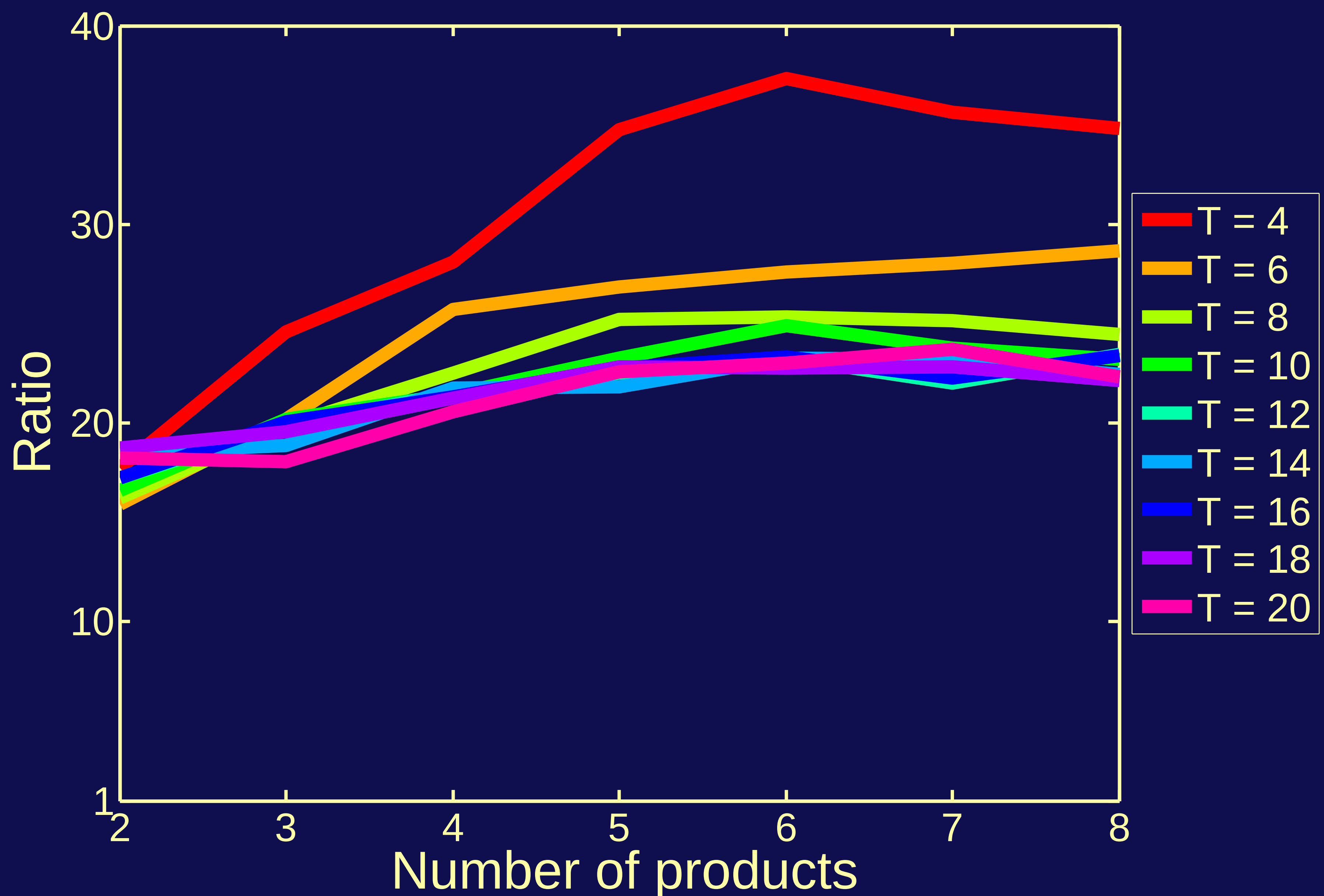
- Takes values between 0 and 1 (both achievable)
- Suitable if $C^{\max}$ can be calculated and is not ∞
- π_1 is better than π_2 following $\text{rsg} \equiv$
  π_1 is better than π_2 following arsg
- Interpretation: Fraction of absolute gap $C^{\max} - C^{\min}$
  that is not avoided

Other Heuristics

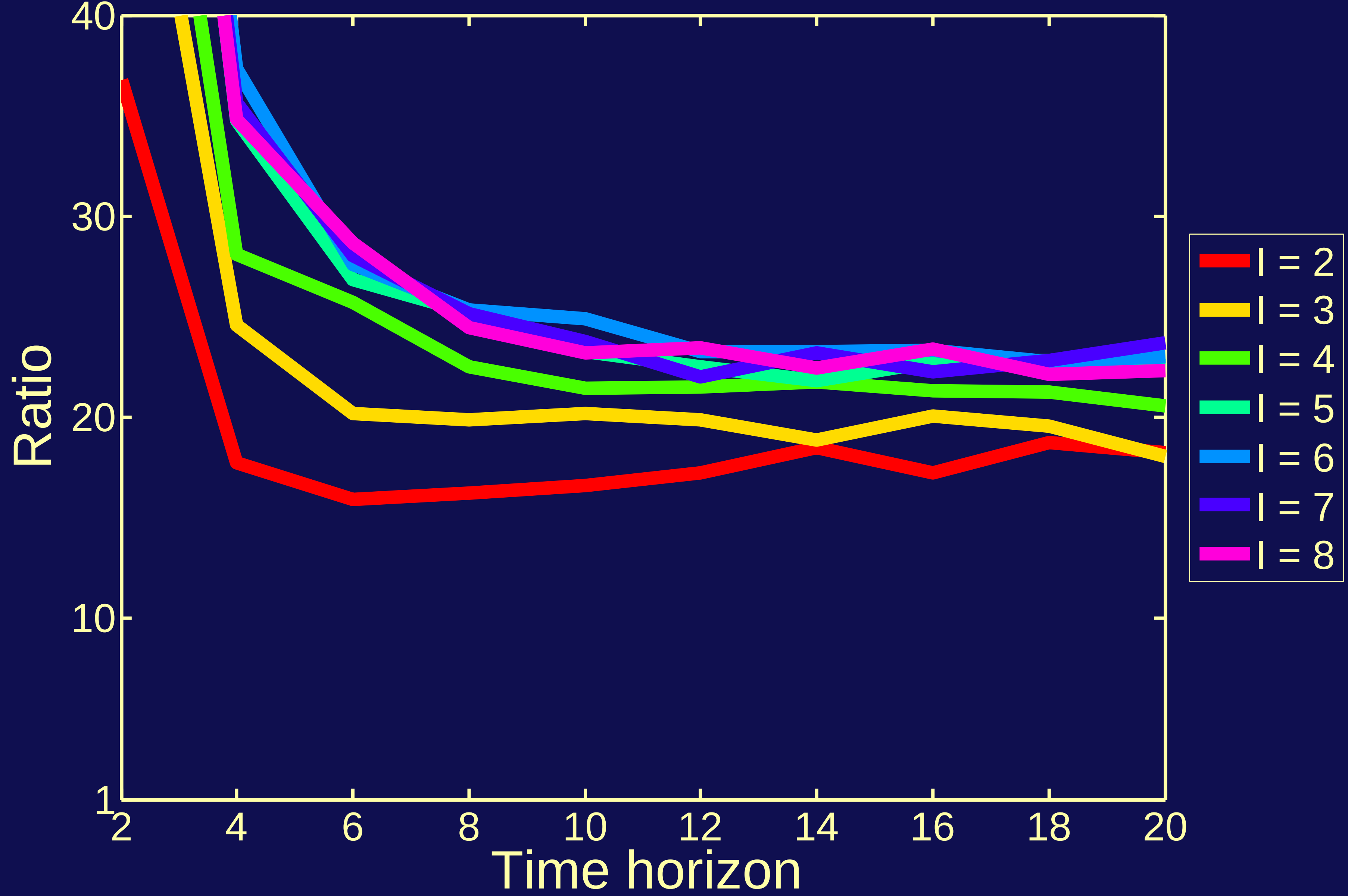
- EDF policy: Products with Earlier Deadline go First
  - ▷ naive benchmark policy
- GRE policy: Solving (KP) by greedy heuristic
  - ▷ to be used when (KP) is computationally intractable
  - ▷ based on Niño-Mora (2002)
- Define performance ratio of policy π

$$\mathrm{ratio}(\pi) = \frac{\mathrm{mean}\left(\mathrm{arsg}(\pi)\right)}{\mathrm{mean}\left(\mathrm{arsg}(\mathsf{PI})\right)}$$

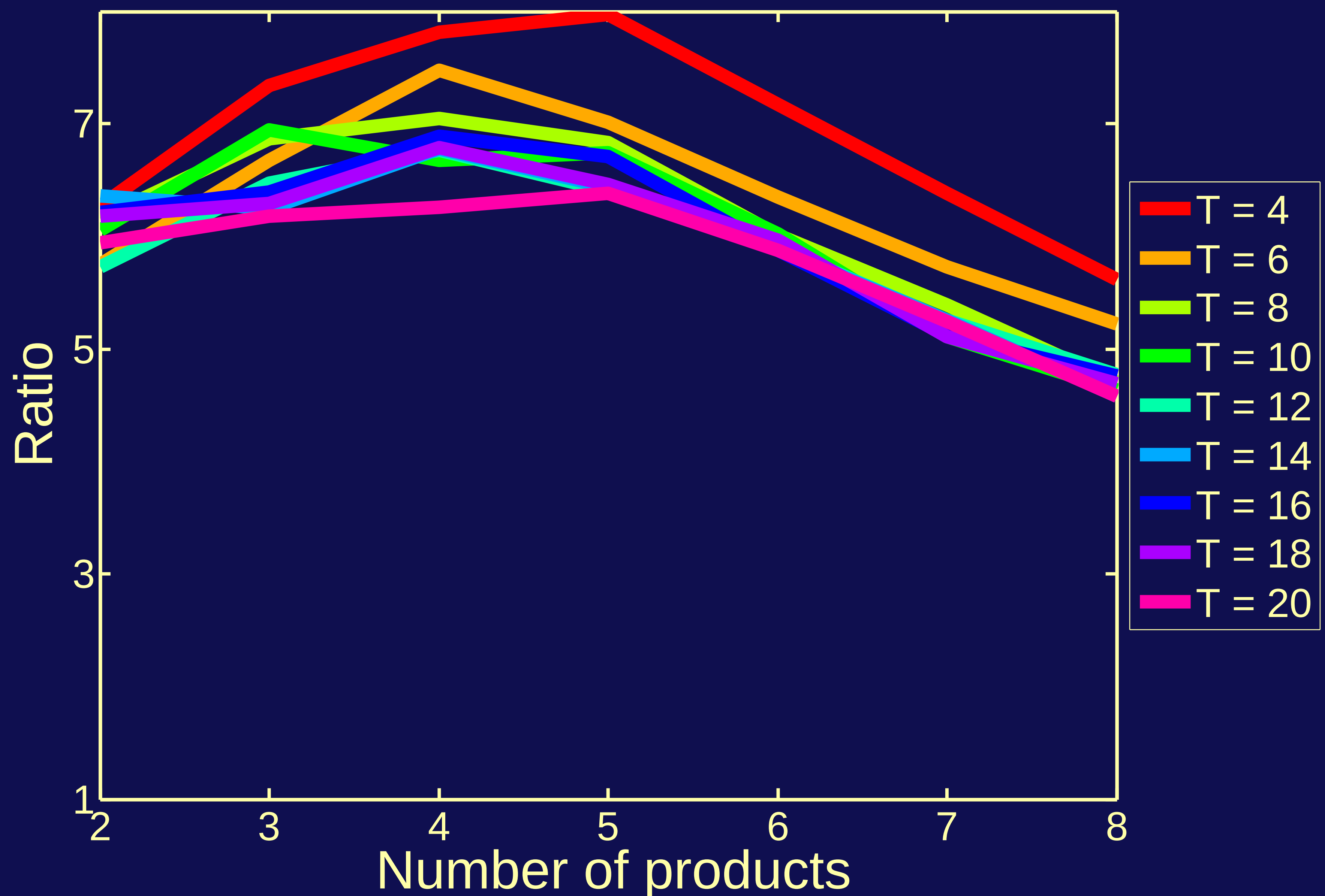
Performance Ratio of EDF Policy



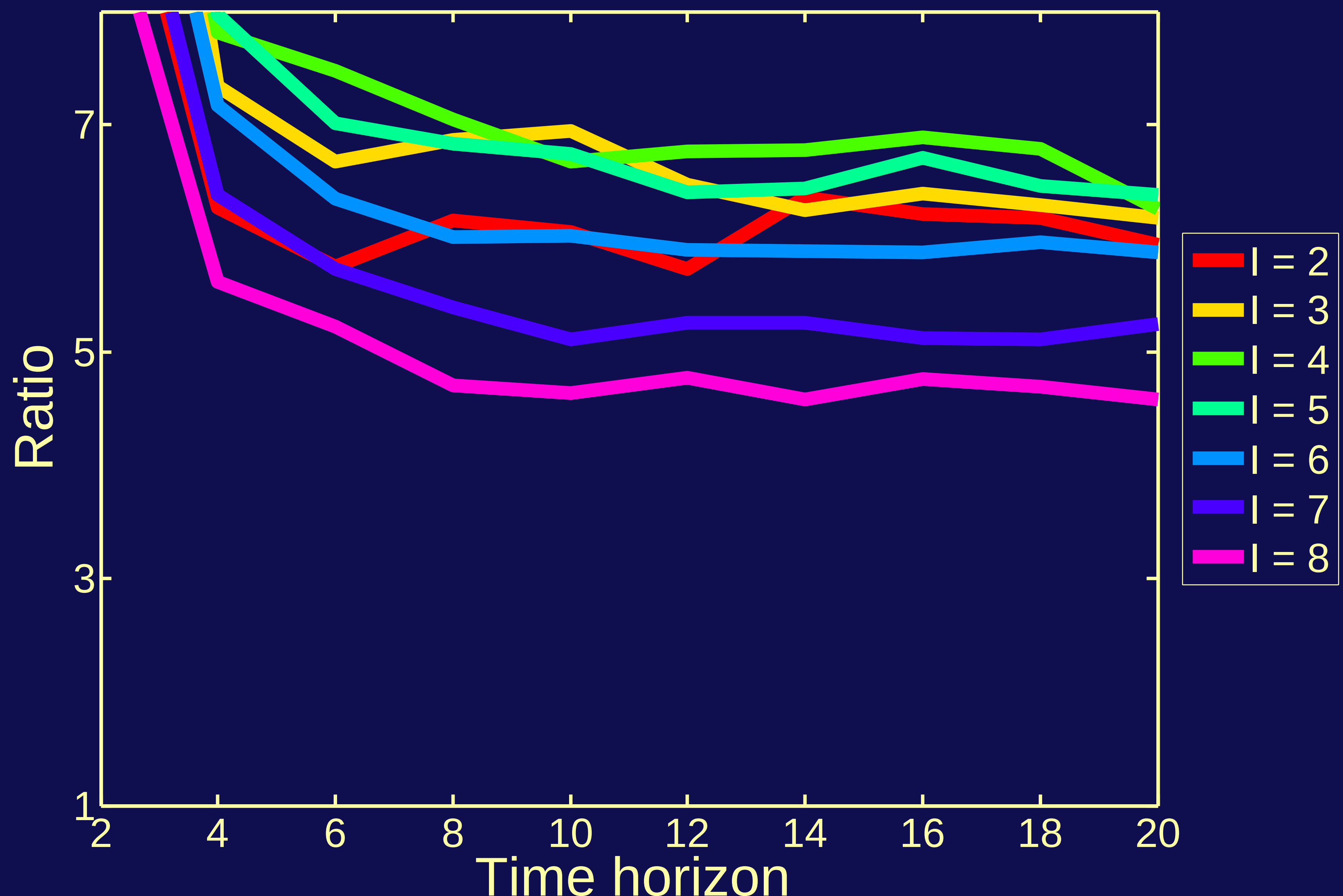
Performance Ratio of EDF Policy



Performance Ratio of GRE Policy



Performance Ratio of GRE Policy



Summary

- We have presented:
  - ▷ a nontrivial problem with closed-form PI
  - ▷ an optimal promotion policy for perishable items
  - ▷ a new index-knapsack heuristic achieving nearly-optimal performance for KPPI
  - ▷ applicable to a variety of ad-hoc restrictions
  - ▷ new policy performance measure for bounded problems
- What to do: inventory control

Thank you for your attention