Knapsack Problem for Perishable Inventories

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Motivation

$\rightarrow$ net revenue maximization

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products

- why perishable?
    - perishable goods - food, change in fashion, design

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  - lower the price of those products

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    or
  - promote those products

Problem

who	boy	retailer
what to do	go camping	sell the inventories
limits	bag	promotion space (shelf, room)
what	clothes and other camping stuff	products (inventories)
priority	most necessary things	maximization of the revenue

Outline of the Presentation

- formulate the problem in the framework of Markov decision processes (MDP) with a sample-path knapsack capacity constraint
- formulate the KPPIs problem
- apply Whittle relaxation and Lagrangian method, and decompose the problem

! derive the index

! introduce *Index Knapsack* heuristic and its performance and show the near-optimality

Knapsack Properties

- I perishable products $i \in \mathcal{I}$
- K_i units of each product i
- H planning horizon, $H \leq \infty$
- C knapsack volume, $\sum_i W_i > C$, where W_i is volume of every unit of product i

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- $R_i > 0$ revenue
- $\alpha_i R_i$ salvage revenue, where $\alpha_i \leq 1$

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- β discount factor

Demand

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$\rightarrow$ increase: $q_i - p_i > 0$

Formulation of MDP Model of Perishable Inventory with Bernoulli Demand

- state $\boxed{n = (t, k)}$, where
  - t represents the number of remaining periods before the deadline, and
  - k represents the remaining inventory;

  state $\boxed{n = 0}$ - product is perished or there are no units left

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- action $\boxed{a}$
  - 1 to promote a unit
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- action $\boxed{a}$
  - 1 to promote a unit
  - 0 not to promote;
- expected one-period capacity occupation:

$$W_{i,n}^{a} := \begin{cases} W_i & a = 1 \\ 0 & a = 0 \end{cases}$$

- one-period transition probability matrix $\mathbf{P_i^{1|\mathcal{N}_i}}$ under promoting (for $K_i = 2$)

	0	$(1,1)$	$\dots$	$(T_i-1,1)$	$(T_i,1)$	$(1,2)$	$\dots$	$(T_i-1,2)$	
0	1	0	0	0	0	0	0	0	0
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$\mathbf{P_i^{0|\mathcal{N}_i}}$ by substituting p_i by q_i in $\mathbf{P_i^{1|\mathcal{N}_i}}$

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- expected one-period revenue:

$$R^1_{i,(t,k)} := R_i(1-p_i)$$
$$R^1_{i,(1,k)} := R_i(1-p_i) + \beta\alpha_i R_i(p_i + k - 1)$$
$$R^1_{i,0} := 0$$

KPPIs, Relaxation and Decomposition

$$\max_{\boldsymbol{\pi}\in\Pi_{\mathbf{X},\mathbf{a}}} \mathbb{E}_0^{\boldsymbol{\pi}} \left[\sum_{i\in\mathcal{I}} \sum_{s=0}^{H} \beta^s R_{i,X_i(s)}^{a_i(s)} \right]$$

$$\text{subject to} \quad \sum_{i\in\mathcal{I}} W_{i,X_i(s)}^{a_i(s)} \leq C \text{ at each time period } s \in \mathcal{H},$$

where $\mathbf{X}(\cdot) := (X_i(\cdot))_{i\in\mathcal{I}}$ is the joint state-process; and
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one product parametric optimization subproblem:

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Indexability

Definition (Indexability).

We say that ν-parameterized inventory is *indexable*, if there exist unique values $-\infty \le \nu_n \le \infty$ for all $n \in \mathcal{N}$ such that the following holds:

1. if $\nu_n \ge \nu$, then it is optimal to promote in state n, and
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- MDP theory $\rightarrow$ existence of an optimal policy (stationary, deterministic, independent on initial state)
- $\mathcal{S}(\nu)$ active set representing a stationary policy, set of all states with action 1

Analytical approach

provably indexable products

Numerical approach

numerical testing of indexability

Provably Indexable Products

Assumption

1. $q - p > 0$, $\alpha \leq 0$ for $\beta \leq 1$.
2. family $\mathcal{F}_1$ of active sets

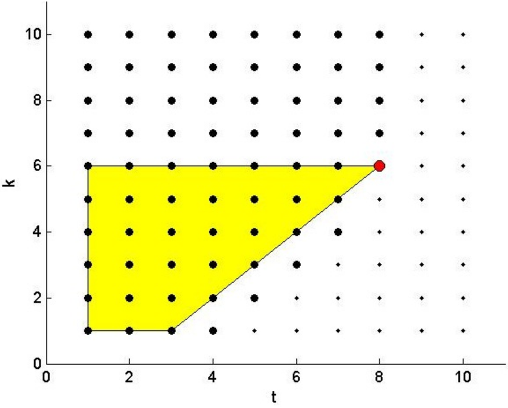

Figure: Behavior of added states ($\mathcal{S}$ filled area).

Theorem (Indexability).

If for every ν there is an optimal active set that belongs to $\mathcal{F}_1$, then the product is indexable, and the index value for its state $(t,k) \in \mathcal{T} \times \mathcal{K}$ is

$$\nu^*_{(t,k)} = \begin{cases} \dfrac{R}{W}(1-p)\left[1 - \dfrac{1-q+(q-p)\beta^t\alpha}{1-p}\right] & t \leq k \\ \dfrac{R}{W}(1-p)\left[1 - \dfrac{1-q+\beta^t\alpha(q-p)p^{t-k}\sum\limits_{i=0}^{k-1}\binom{t-k-1+i}{i}(1-p)^i}{1-p-(q-p)\beta^k(1-p)^k\sum\limits_{i=0}^{t-k-1}\binom{k-1+i}{i}(\beta p)^i}\right] & t > k \end{cases}$$

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- fast recursive computation: $\mathcal{O}(TK)$

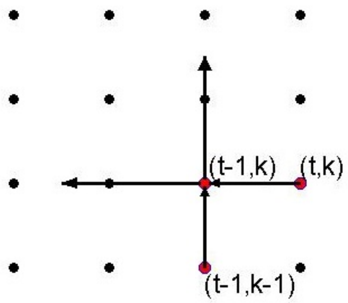
Proposition (Monotonicity).

the monotonicity properties of index are

(i) $\nu^*_{(t-1,k)} \geq \nu^*_{(t,k)}$ $\forall k \geq 1, \forall t > 1$

(ii) $\nu^*_{(t,k-1)} \leq \nu^*_{(t,k)}$ $\forall k > 1, \forall t \geq 1$

(iii) $\nu^*_{(s,l)} \geq \nu^*_{(t,k)}$ $\forall l \geq k, \forall s \leq t$



Numerical Testing

- problem is indexable for all products parameters

Numerical Testing

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- for $\boxed{\alpha \leq 0 \text{ and } \beta \leq 1}$ and for $\boxed{\alpha \leq 1 \text{ and } \beta = 1}$

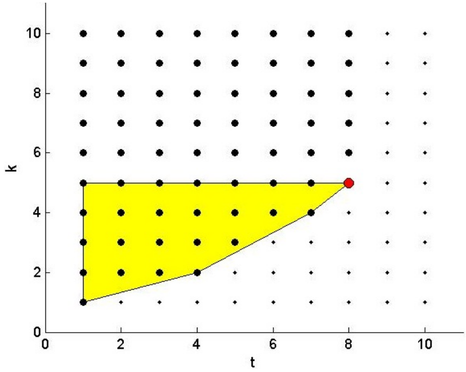

Figure: Behavior of added states ($\mathcal{S}$ filled area).

Index Rule for KPPIs

procedure

- to use the index as a price $v_i^{(s)} := W_i \nu^*_{i,X_i(s)}$, in every $s \in [0, H]$
- to solve the following knapsack problem

$$\begin{aligned} \max_{\mathbf{z}} \sum_{i \in \mathcal{I}} & z_i^{(s)} v_i^{(s)} \\ \text{s.t.} \quad \sum_{i \in \mathcal{I}} & z_i^{(s)} W_i \leq C \\ & z_i^{(s)} \in \{0, 1\} \text{ for all } i \in \mathcal{I} \end{aligned} \tag{KP}$$

where $\mathbf{z}^{(s)} = (z_i^{(s)} : i \in \mathcal{I})$ is vector of binary decision variables.

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$$z_i^{(s)} \in \{0, 1\} \text{ for all } i \in \mathcal{I}$$

where $\mathbf{z}^{(s)} = (z_i^{(s)} : i \in \mathcal{I})$ is vector of binary decision variables.

(IK) *Index-Knapsack heuristic:* Calculate the prices v_i and then solve the knapsack problem optimally.

Suboptimality

solving KPPIs

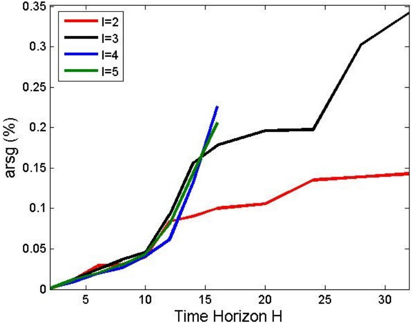
- optimally $\rightarrow D^{\max}$
- by employing the heuristic $\rightarrow D^{\pi}$

adjusted relative suboptimality gap

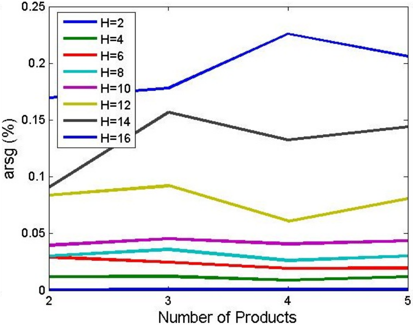
$$\operatorname{arsg}(\pi) = \frac{D^{\max} - D^{\pi}}{D^{\max} - D^{\min}},$$

where $0 \leq \operatorname{arsg}(\pi) \leq 1$

Results for Index-Knapsack heuristic
(analytically computed index)



(a) for values $H = 2, 4, \ldots, 16$



(b) for values $I = 2, 3, 4, 5$

Figure: Mean adjusted relative suboptimality gap for IK heuristic with analytically computed index.

Conclusion

- formulation of the problem as MDP
- discussion of the optimal policy from analytical and numerical point of view
- derivation of the index
- showing the near-optimality of IK heuristic

Thank you for your attention.