An Optimal Index Policy for the Multi-Armed Bandit Problem with Re-Initializing Bandits

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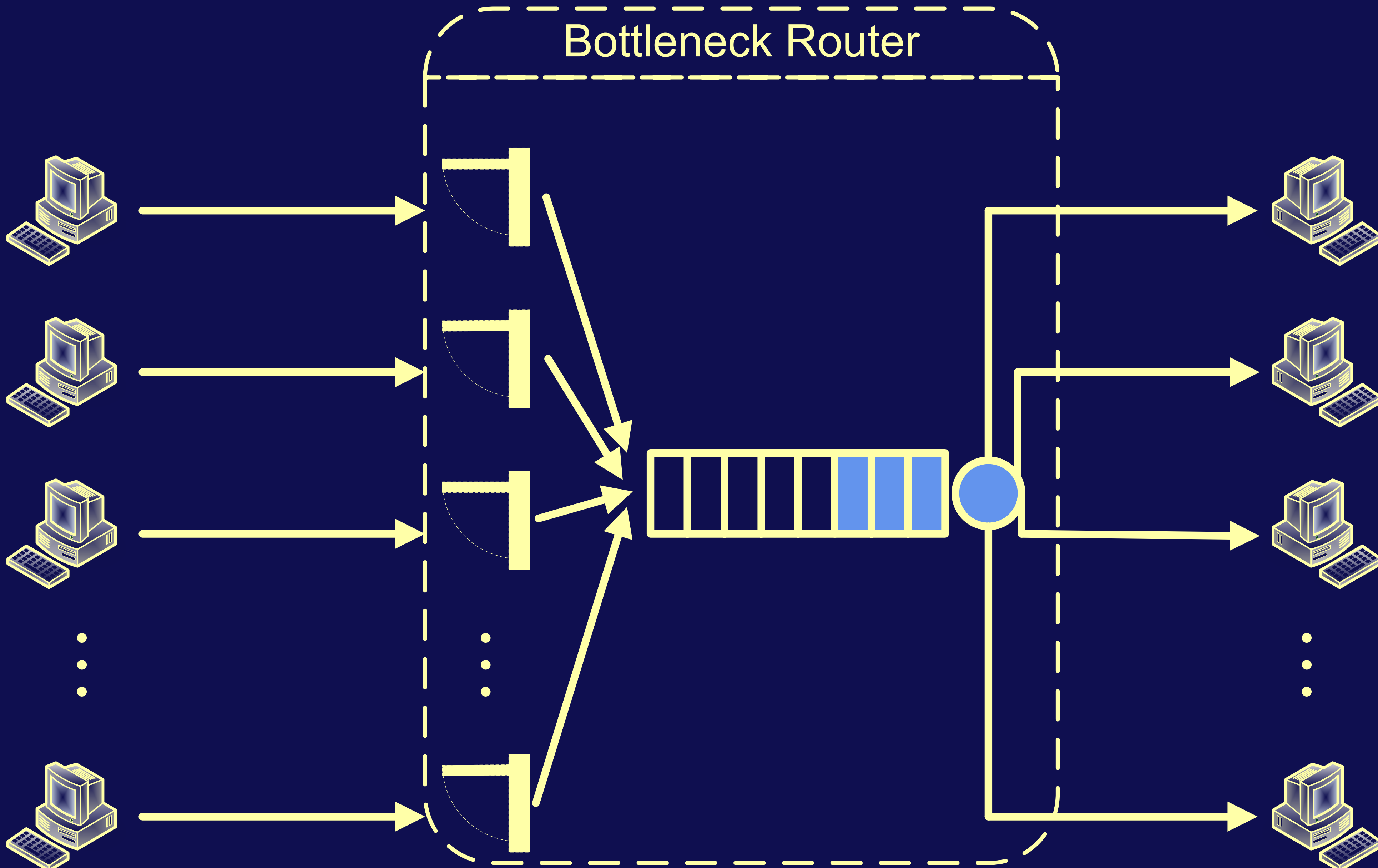
Example: Congestion Control in Router



Multi-Armed Bandit Problem

Multi-Armed Bandit Problem

- A classic problem of efficient learning
- Originally in sequential design of experiments
  - ▷ Thompson (1933): which of two drugs is superior?
  - ▷ Robbins (1952), Bradt et. al (1956), Bellman (1956)
- Job sequencing problem
  - ▷ Cox & Smith (1961): $c\mu$-rule
- Celebrated general solution
  - ▷ Gittins and colleagues (1970s): Gittins index rule
  - ▷ crucial condition: non-played are frozen

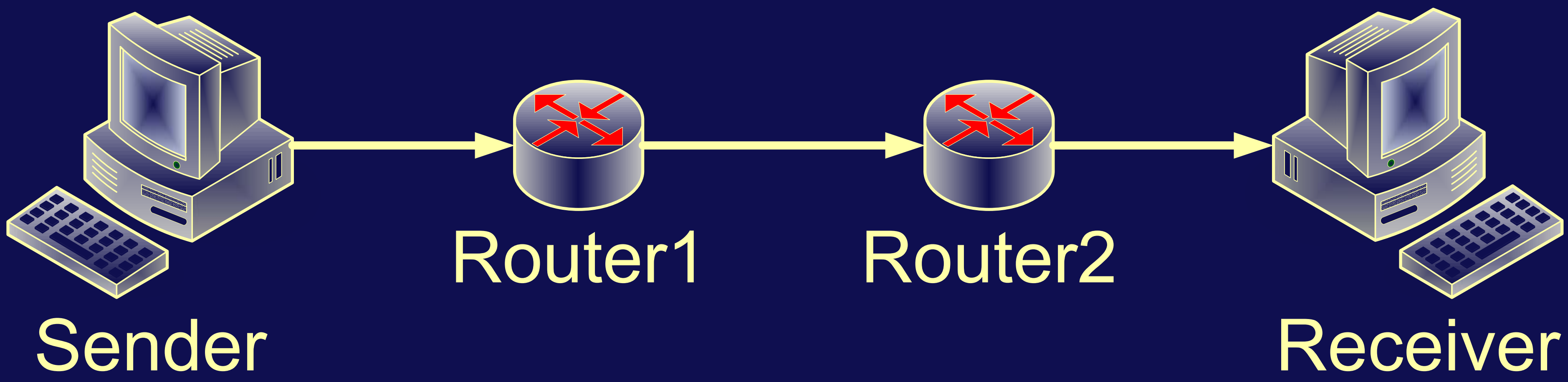
Outline

- Re-initializing bandits
  - ▷ Transmission Control Protocol (TCP)
- MDP formulation
- Relaxations and decomposition into subproblems
- Optimal solution to the subproblems
  - ▷ obtaining an index
- Optimal solution to relaxations
- Optimal index policy to original problem

Transmission Control Protocol (TCP)

- Implemented at the two ends of a connection
- A way of end-to-end congestion control
- Provides reliable, ordered delivery of a stream of packets
- Fully-sensitive to packet losses
- Examples: web browsers, e-mail, file transfer (FTP)
- Must be distinguished from congestion control in routers
- An alternative (UDP) is used for VoIP, streaming, etc.

TCP End-to-End Connection



- Sender sends an initial packet and waits
- Receiver sends acknowledgment of each received packet
- Sender sends more packet(s) after receiving acknowledgement(s) or restarts after time-out

TCP Dynamics as Markov Chain

- States: $n \in \{0, 1, \ldots, N-1\}$ = sending rate level
  - ▷ $n = 0$: sending rate of 1 packet/RTT
  - ▷ $n = N-1$: maximum rate, $\leq W^{\max}$
- Transitions: OK (acknowledgment), NO (time-out)

Congestion Control of TCP Flows in Router

- Time epochs $t = 0, 1, 2, \dots$
- Two possible control actions $a(t)$:
  - ▷ transmit the flow packets
  - ▷ block the flow by dropping packets
- If transmitted $W_{X(t)}$ packets in state $X(t)$, then
  - ▷ goodput (reward) $R_{X(t)}$ is earned
  - ▷ the sender sets $X(t+1)$ given by TCP dynamics
- Objective: Maximize the long-run goodput (reward)
  - ▷ while choosing exactly one flow every time epoch

Congestion Control in Router

- Maximizing time-average expected goodput

$$\max_{\pi \in \Pi} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{\boldsymbol{n}}^{\pi} \left[\sum_{t=0}^{T-1} \sum_{m \in \mathcal{M}} R_{m, X_m(t)}^{a_m(t)} \right]$$

- Subject to sample path condition

$$\sum_{m \in \mathcal{M}} a_m(t) = 1, \ \text{for all } t$$

conditional on state history under π

Relaxations

- 1: Whittle's Relaxation: choose one on average

$$\lim_{T\to\infty} \frac{1}{T} \mathbb{E}_{\boldsymbol{n}}^{\pi} \left[\sum_{t=0}^{T-1} \sum_{m\in\mathcal{M}} a_m(t) \right] = 1$$

- 2: Multiply by $W^{\max}$ and use $W_{m,X_m(t)}^{a_m(t)} \leq W^{\max} a_m(t)$,

$$\lim_{T\to\infty} \frac{1}{T} \mathbb{E}_{\boldsymbol{n}}^{\pi} \left[\sum_{t=0}^{T-1} \sum_{m\in\mathcal{M}} W_{m,X_m(t)}^{a_m(t)} \right] \leq W^{\max}$$

- 3: Dualize this constraint using Lagrangian multiplier

$$\max_{\pi\in\Pi} \lim_{T\to\infty} \frac{1}{T} \mathbb{E}_{\boldsymbol{n}}^{\pi} \left[\sum_{t=0}^{T-1} \sum_{m\in\mathcal{M}} \left(R_{m,X_m(t)}^{a_m(t)} - \nu W_{m,X_m(t)}^{a_m(t)} \right) \right] + \nu W^{\max}$$

Decomposition

- Decompose the Lagrangian relaxation due to flow independence into single-flow parametric subproblems

$$\max_{\pi_m \in \Pi_m} \lim_{T\to\infty} \frac{1}{T} \mathbb{E}^{\pi_m}_{n_m} \left[\sum_{t=0}^{T-1} \left(R^{a_m(t)}_{m,X_m(t)} - \nu W^{a_m(t)}_{m,X_m(t)} \right) \right]$$

- This is known as a restless bandit

- Under certain natural conditions, there exist break-even values $\nu_{m,n}$ of ν, called transmission indices (prices), s.t.
  - ▷ it is optimal to transmit if $\nu_{m,n} \geq \nu$
  - ▷ it is optimal to block if $\nu_{m,n} \leq \nu$

Optimal Solutions to Relaxations

- For 3 (multi-flow Lagrangian relaxation): For each flow,
  - ▷ it is optimal to transmit if $\nu_{m,n} \geq \nu$
  - ▷ it is optimal to block if $\nu_{m,n} \leq \nu$
- Suppose that re-initializing state 0 has highest value, i.e., $\nu_{m,0} \geq \nu_{m,n}$ for all states n of any flow m
- Denote by ν^* the second-highest $\nu_{m,0}$ over m
- For 1: an optimal policy is: at every t,
  - ▷ transmit each flow satisfying $\nu_{m,X(t)} > \nu^*$
  - ▷ if no such flow exists, then transmit one flow satisfying $\nu_{m,X(t)} = \nu^*$

Optimal Solution to Original Problem

- An optimal policy is: at every t,
  - ▷ transmit each flow satisfying $\nu_{m,n} > \nu^*$
  - ▷ if no such flow exists, then transmit one flow satisfying $\nu_{m,X(t)} = \nu^*$
- This policy chooses exactly one flow every time epoch
- It is optimal here because it is feasible here and optimal for a relaxation
- (See animation)

Multi-Armed Bandit Problem

- We can apply the same reasoning to the classic problem
  - ▷ Set the threshold to the second-highest Gittins index
  - ▷ Play the bandits with Gittins index higher than the threshold, breaking ties choosing one arbitrarily
  - ▷ Once no bandits are above, restart the procedure
- Optimal policy $=$ sequence of optimal solutions to Lagrangian relaxations with decreasing values of Lagrangian multiplier

Routing: A More Realistic Setting

- Bandwidth W, i.e., deterministic "server capacity"
- Target time-average router throughput $\overline{W} < W$, i.e., "virtual capacity"
- Buffer size $B \geq W$
- Backlog process $B(t)$ at epochs t
  - ▷ number of packets buffered for more than one period
- To be allocated to randomly appearing and disappearing flows

Summary

- Apart from multi-armed bandit problem and its special cases, proving optimality of an index policy is rare
- The approach leads to a new proof for the classic problem
- For more complex (restless) bandit problems
  - ▷ gives some intuition for when an index policy is optimal
  - ▷ presents a well-grounded method for design of (suboptimal) greedy rules
  - ▷ useful for problems with on-average constraint (if capacity can be marketed between periods)

Thank you for your attention!

Congestion Control in Router