Some Bounds for Labeling with a Condition at Distance Two

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Graph Coloring Representation of FAP

- Vertices (nodes) stand for frequency transmitters
- Edges (arcs) join pairs of "very close" transmitters
- Graph distance (# edges) is a proxy for the true distance
- Vertex labels (nonnegative integers) represent channels
- Frequency interference problem prescribes coloring restrictions, which can arbitrarily closely approximate real-life situation

Popular Approximations

- *d-distant coloring* (coloring of graph powers)
  - ⋆ Labels of vertices within distance d must differ
- *$L(2,1)$-labeling*
  - ⋆ Labels of adjacent vertices must differ by 2, of those at distance 2 by 1
- *$L(k,1)$-labeling*
  - ⋆ Labels of adjacent vertices must differ by k, of those at distance 2 by 1

$L(k,l)$-labeling

Definition. A *labeling* φ_n associates every vertex of a graph G with one number from the set of labels $\{0, 1, 2, \ldots, n\}$.

Definition. Labeling φ_n is called an $L(k,l)$-*labeling* if

- $|\varphi_n(x) - \varphi_n(y)| \geq k$, whenever x, y are neighbours;
- $|\varphi_n(x) - \varphi_n(y)| \geq l$, whenever x, y are at distance two.

$L(k,l)$-labeling

- It is a generalization of the previous versions (2-distant coloring, $L(2,1)$-labeling and $L(k,1)$-labeling).
- We can WLOG suppose that k and l are nonnegative integers and we **omit** the restriction of $k \geq l$.
- For a given graph G, an $L(k,l)$-labeling φ_n with minimal n is called *optimal* and defines the *$L(k,l)$-labeling number* $\lambda_G(k,l) = n$.

Results

Lemma 1. Let a, b be nonnegative integers such that $a > b$ and m be a positive integer. Then

$$\left\lfloor \frac{a}{m} \right\rfloor - \left\lfloor \frac{b}{m} \right\rfloor > \frac{a-b}{m} - 1, \text{ and also } \left\lfloor \frac{a}{m} \right\rfloor \geq \left\lfloor \frac{b}{m} \right\rfloor.$$

(where $\lfloor x \rfloor$ denotes the lower integer part of x.)

Theorem 2 (Scaling property).
It is $\lambda(\alpha k, \alpha l) = \alpha \cdot \lambda(k, l)$ for any positive integer α.

Proof of Theorem 2 (a sketch)

(i) If φ_n is an optimal $L(k,l)$-labeling, then $\alpha \cdot \varphi_n$ is clearly an $L(\alpha k, \alpha l)$-labeling, so $\lambda(\alpha k, \alpha l) \leq \alpha \cdot \lambda(k,l)$.

(ii) Denote $k_1 = k, k_2 = l$. Suppose φ_n is an optimal $L(\alpha k, \alpha l)$-labeling, we will show that $\psi = \lfloor \frac{\varphi_n}{\alpha} \rfloor$ is an $L(k,l)$-labeling. Cases, if $k_i = 0$ or if for two vertices x, y at distance $i \leq 2$ is $\varphi_n(x) = \varphi_n(y)$, are trivial.

Othewise, suppose WLOG that $\varphi_n(x) > \varphi_n(y)$. Then $\psi(x) - \psi(y) = \lfloor \frac{\varphi_n(x)}{\alpha} \rfloor - \lfloor \frac{\varphi_n(y)}{\alpha} \rfloor > \frac{\varphi_n(x) - \varphi_n(y)}{\alpha} - 1 \geq k_i - 1$, so $\psi(x) - \psi(y) \geq k_i$ and $\lambda(\alpha k, \alpha l) \geq \alpha \cdot \lambda(k,l)$. qed.

Relation with 2-distant Coloring

Corollary 2.1 It is $\lambda(k,k) = k \cdot (\chi_2 - 1)$.

Corollary 2.2 It is
$\min\{k,l\} \cdot (\chi_2 - 1) \leq \lambda(k,l) \leq \max\{k,l\} \cdot (\chi_2 - 1)$.

- These are true, because $\lambda(1,1) = \chi_2 - 1$.
- Theorem 2 and listed consequences also hold in a more dimensional restrictions space

Relation with $L(2,1)$- and $L(1,2)$-labeling

Corollary 2.3 It is $\lambda(k,l) \leq \max\{\lfloor\frac{k+1}{2}\rfloor, l\} \cdot \lambda(2,1)$ and $\lambda(k,l) \leq \max\{k, \lfloor\frac{l+1}{2}\rfloor\} \cdot \lambda(1,2)$.

Theorem 3.

(a) If $k \geq 2l$, then $\lambda(k,l) \leq \left\lfloor \frac{\lambda(2,1)}{2} \right\rfloor k + \left\{ \frac{\lambda(2,1)}{2} \right\} 2l$.

(b) If $2k \leq l$, then $\lambda(k,l) \leq \left\{ \frac{\lambda(1,2)}{2} \right\} 2k + \left\lfloor \frac{\lambda(1,2)}{2} \right\rfloor l$.

(where $\{x\} = x - \lfloor x \rfloor$, so it is $0 \leq \{x\} < 1$.)

Proof of Theorem 3 (a sketch)

(a) If φ_n is an optimal $L(2,1)$-labeling, then it is enough to show that $\psi = \left\lfloor \frac{\varphi_n}{2} \right\rfloor k + \left\{ \frac{\varphi_n}{2} \right\} 2l$ is an $L(k,l)$-labeling.

If for two vertices x, y it was $\varphi_n(x) = \varphi_n(y) + 1$, then $\psi(x) - \psi(y) \geq l$. Moreover, $\varphi_n(x) = \varphi_n(y) + 2$ implies $\psi(x) - \psi(y) = k$. Since ψ is nondecreasing as function of φ_n, the proof is finished.

(b) Proceed analogously. qed.

Upper Bounds (1)

Theorem 4.
It is $\lambda(k,l) \leq (2l-1)\Delta^2 + 2(k-l)\Delta$.

(where Δ is the maximum degree of the graph.)

The proof of Theorem 4 is done by greedy labeling, analogously to that presented by Griggs & Yeh (1992).

Upper Bounds (2)

Theorem 5.

(a) If $k > l$, then $\lambda(k,l) < l\Delta^2 + k\Delta$. Moreover, if $\frac{k}{l}$ is an integer, then $\lambda(k,l) \leq l\Delta^2 + (k-l)\Delta$.

(b) If $k \leq l$, then $\lambda(k,l) \leq l\Delta^2 + l\Delta$.

The proof of Theorem 5 draws on the idea of the proof of the bound $\Delta^2 + \Delta$ on $L(2,1)$-labeling number, introduced in Chang & Kuo (1996).

Lower Bounds (1)

Theorem 6. For a nontrivial connected graph it is $\lambda(k,l) \geq k + (\delta - 1)l$.

(where δ is the minimum degree of the graph.)

PROOF: Look at any vertex labeled by 0 (such a vertex exists in any optimal $L(k,l)$-labeling) with all its neighbours. As the mutual distances of its neighbours are 2, the span of their labels must be at least $(\delta - 1)l$. Moreover, the lowest label cannot be lower than k, as their are neighbours of the vertex labeled by 0. qed.

Lower Bounds (2)

Theorem 7. Let the maximum degree Δ be at least 2.

(i) It is

$$\lambda(k,l) \geq \begin{cases} k + (\Delta - 1)l, & \text{if } l \leq k \\ 2k + (\Delta - 2)l, & \text{if } k \leq l \leq 2k \\ (\Delta - 1)l, & \text{if } 2k \leq l \end{cases}$$

(ii) If $k > l$ and $\lambda(k,l) = k + (\Delta - 1)l$, then every vertex of degree Δ is labeled by 0 or by $k + (\Delta - 1)l$ in all optimal $L(k,l)$-labelings.

Proof of Theorem 7 (a sketch)

(i) Consider the vertex x whose degree is Δ. If all its neighbours are labeled by a label larger than x's label, or their labels are all lower than x's label, then by the same argument as in the proof of Theorem 6, we need to use at least the label $k + (\Delta - 1)l$.

Otherwise, one can show that it is needed either the label $(\Delta - 1)l$ or the label $2k + (\Delta - 2)l$. By considering the minimum of all these values we reach the statement of the theorem.

Proof of Theorem 7 (a sketch)

(ii) One can show that, in this case, the neighbours of x have labels either all larger or all lower than x's label. qed.

Note that Theorem 7 is a generalization of items (i) and (ii) of Lemma 2.1 presented in Whittlesey, Georges, & Mauro (1995), which only dealt with $L(2,1)$-labeling.

Thank you!