Some Bounds for Labeling with a Condition at Distance Two[1]
(Extended Abstract)

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Frequency assignment is a classical problem in telecommunications (Hale 1980). A natural mathematical setting, which can be used to approximate real-life problems arbitrarily closely, is graph coloring. Different approximations have been studied extensively, including distance coloring (e.g. Heuvel & McGuinness 2003) and $L(2,1)$-labeling (e.g. Griggs & Yeh 1992). A more realistic modelling is possible using a more general $L(k,l)$-labeling (Georges & Mauro 1995). In this paper we present a list of bounds for optimal $L(k,l)$-labeling of graphs.

A labeling φ_n associates every vertex of a graph G with one number from the set of labels $\{0,1,2,\dots,n\}$. If φ_n is such that the absolute-value difference of labels of any pair of adjacent vertices is at least k and the absolute-value difference of labels of any pair of vertices at distance two is at least l, then we call it an $L(k,l)$-labeling. It is not difficult to see that it is not restrictive to suppose that k and l are nonnegative integers. If φ_n is $L(k,l)$-labeling of graph G and there is no $L(k,l)$-labeling ψ_m of this graph with $m < n$, we consider φ_n to be optimal $L(k,l)$-labeling of G and define $L(k,l)$-labeling number $\lambda_G(k,l) = n$. In what follows, everything relates to an arbitrary but fixed graph G.

Let us first introduce some notation. Let $\lfloor x \rfloor$ for any real number x denote lower integer part of x, i.e. $\lfloor x \rfloor$ is an integer such that $x - 1 < \lfloor x \rfloor \le x$. Denote also $\{x\} = x - \lfloor x \rfloor$, so it is $0 \le \{x\} < 1$.

Lemma 1 Let a, b be nonnegative integers such that $a > b$ and m be a positive integer. Then

$$\left\lfloor \frac{a}{m} \right\rfloor - \left\lfloor \frac{b}{m} \right\rfloor > \frac{a-b}{m} - 1, \text{ and also } \left\lfloor \frac{a}{m} \right\rfloor \ge \left\lfloor \frac{b}{m} \right\rfloor .$$

Theorem 2 It is $\lambda(\alpha k, \alpha l) = \alpha \cdot \lambda(k,l)$ for any positive integer α.

PROOF: Let $f(\varphi_n)$, for any nonnegative-integer-valued function f, be a labeling, which associates all the vertices with label $f(\cdot)$ instead of the label assigned by the labeling φ_n. Given an optimal $L(k,l)$-labeling φ_n, it is straightforward that $\alpha \cdot \varphi_n$ is an $L(\alpha k, \alpha l)$-labeling, and so it must be $\lambda(\alpha k, \alpha l) \le \alpha \cdot \lambda(k,l)$.

On the other hand, using Lemma 1 it is possible to prove that, given an optimal $L(\alpha k, \alpha l)$-labeling φ_n, mapping $\lfloor \frac{\varphi_n}{\alpha} \rfloor$ is an $L(k,l)$-labeling, which implies that $\lambda(\alpha k, \alpha l) \ge \alpha \cdot \lambda(k,l)$. □

Corollary 2.1 It is $\lambda(k,k) = k \cdot \lambda(1,1)$.

Corollary 2.2 It is $\min\{k,l\} \cdot \lambda(1,1) \le \lambda(k,l) \le \max\{k,l\} \cdot \lambda(1,1)$.

Corollary 2.2 comes from an observation that the larger k and l are, the more restricted the problem is, and thus $\lambda(k,l) \le \lambda(k^*, l^*)$ whenever $k \le k^*$ and $l \le l^*$. Both Corollary 2.1 and Corollary 2.2 are important

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from the point of view that they relate $L(k,l)$-labeling number to distance chromatic number $\chi_2 = \lambda(1,1)+1$. Indeed, $L(1,1)$-labeling problem is equivalent to coloring problem of the square of the graph, which is in turn equivalent to 2-distance graph coloring.

Another consequence, stated in Corollary 2.3, is a relation with a simpler and well studied $L(2,1)$-labeling number. We also consider a bound that uses $L(1,2)$-labeling number, which may for some graphs be easier to obtain than $L(2,1)$-labeling number. Theorem 3 then indicates cases, when a sharper bound for $L(k,l)$-labeling number by an expression of $\lambda(2,1)$ or $\lambda(1,2)$ exists. Its proof can be found in Jacko (2003).

Corollary 2.3 It is $\lambda(k,l) \leq \max\{\lfloor \frac{k+1}{2} \rfloor, l\} \cdot \lambda(2,1)$ and $\lambda(k,l) \leq \max\{k, \lfloor \frac{l+1}{2} \rfloor\} \cdot \lambda(1,2)$.

Theorem 3

(a) If $k \geq 2l$, then $\lambda(k,l) \leq \left\lfloor \dfrac{\lambda(2,1)}{2} \right\rfloor k + \left\{ \dfrac{\lambda(2,1)}{2} \right\} 2l.$

(b) If $2k \leq l$, then $\lambda(k,l) \leq \left\{ \dfrac{\lambda(1,2)}{2} \right\} 2k + \left\lfloor \dfrac{\lambda(1,2)}{2} \right\rfloor l.$

Up to this point we have presented relationships between $L(k,l)$-labeling and other types of colorings and labelings. Nevertheless, such information may not always be available. The rest of the analysis thus expounds various bounds for $L(k,l)$-labeling number employing fundamental graph characteristics: the minimum degree δ and the maximum degree Δ. First we present two upper bounds. In Griggs & Yeh (1992) it was shown that $\lambda(2,1) \leq \Delta^2 + 2\Delta$; analogously, it is possible to get a conclusion stated in Theorem 4. Theorem 5 draws on the idea of the proof of a sharper bound, $\lambda(2,1) \leq \Delta^2 + \Delta$, introduced in Chang & Kuo (1996).

Theorem 4 It is $\lambda(k,l) \leq (2l-1)\Delta^2 + 2(k-l)\Delta$.

Theorem 5

(a) If $k > l$, then $\lambda(k,l) < l\Delta^2 + k\Delta$. Moreover, if $\frac{k}{l}$ is an integer, then $\lambda(k,l) \leq l\Delta^2 + (k-l)\Delta$.

(b) If $k \leq l$, then $\lambda(k,l) \leq l\Delta^2 + l\Delta$.

Theorem 6 and Theorem 7 conclude our list by providing two lower bounds for $L(k,l)$-labeling number. Note that Theorem 7 is a generalization of items (i) and (ii) of Lemma 2.1 presented in Whittlesey, Georges, & Mauro (1995), which only dealt with $L(2,1)$-labeling.

Theorem 6 For a nontrivial connected graph it is $\lambda(k,l) \geq k + (\delta - 1)l$.

Theorem 7 Let the maximum degree Δ be at least 2.

(i) It is

$$\lambda(k,l) \geq \begin{cases} k + (\Delta-1)l, & \text{if } l \leq k \\ 2k + (\Delta-2)l, & \text{if } k \leq l \leq 2k \\ (\Delta-1)l, & \text{if } 2k \leq l \end{cases}$$

(ii) If $k > l$ and $\lambda(k,l) = k + (\Delta-1)l$, then every vertex of degree Δ is labeled by 0 or by $k + (\Delta-1)l$ in all optimal $L(k,l)$-labelings.

References

[1] Chang, G. J. & Kuo, D. (1996): The L(2,1)-labeling Problem on Graphs, *SIAM Journal of Discrete Mathematics* 9, 309-316.

[2] Georges, J. P. & Mauro, D. M. (1995): Generalized Vertex Labelings with a Condition at Distance Two, *Congressus Numerantium* 109, 141-159.

[3] Griggs, J. R. & Yeh, R. K. (1992): Labelling Graphs with a Condition at Distance 2, *SIAM Journal of Discrete Mathematics* 5, 586-595.

[4] Hale, W. K. (1980): Frequency Assignment: Theory and Applications, *Proceedings of IEEE* 68, 1497-1514.

[5] Heuvel, J. van den & McGuinness, S. (2003): Coloring the Square of a Planar Graph, *Journal of Graph Theory* 42, 110-124.

[6] Jacko, P. (2003): Dištančné Farbenia Grafov (Distance Labelings of Graphs), Mgr. Thesis, P. J. Šafárik University.

[7] Whittlesey, M. A., Georges, J. P., & Mauro, D. W. (1995): On the λ-number of Q_n and Related Graphs, *SIAM Journal of Discrete Mathematics* 8 (4), 499-506.