Arc Routing Problems

Solving small-scale
vehicle routing problems
for waste collection

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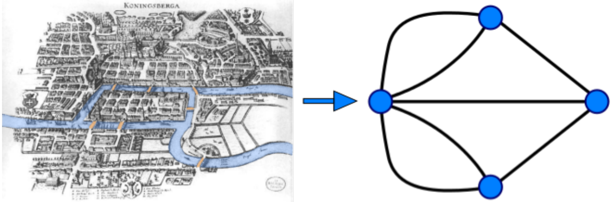
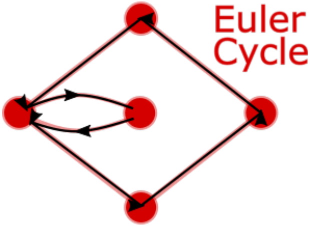
Structure

- Graph Theory - Euler Cycles
- Types of Arc Routing Problems
- Complexity Classes
- Exact solutions for the Chinese Postman Problem
- Integer Linear Programming and Optimisation
- Solutions in C#
- Further Areas of Interest

Königsberg Bridge Problem

Problem: Find a walk through the town that crosses each bridge exactly once.
Euler's Theorem: A connected undirected graph has an Euler cycle if and only if every vertex has even degree.

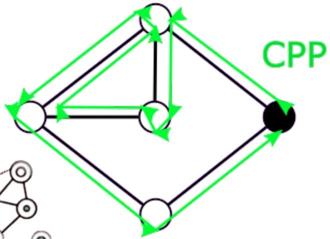




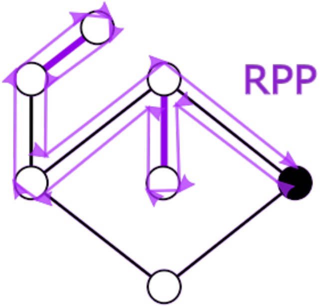
Arc Routing Problems

- find a set of cycles that cover all required edges $R \subseteq E$
- start and end at depot vertex
- minimize the total distance travelled

Chinese Postman Problem
- 1 cycle
- $R = E$



Rural Postman Probem
- 1 cycle
- $R \subset E$

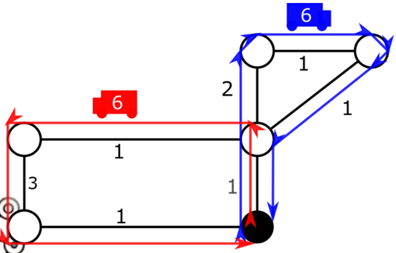


Capacitated Arc Routing

- find a number of vehicle tours
- each vehicle capacity *W*
- satisfy demand for each edge

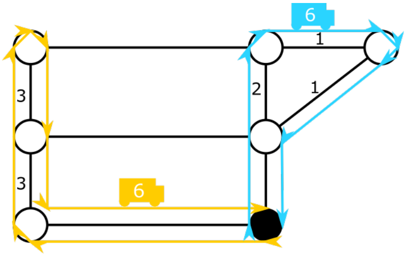
Capacitated CPP

- edge demand > 0

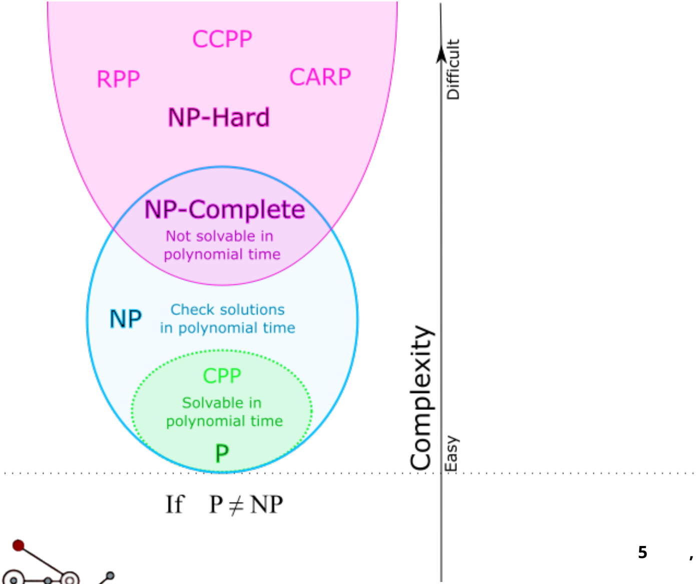


CARP

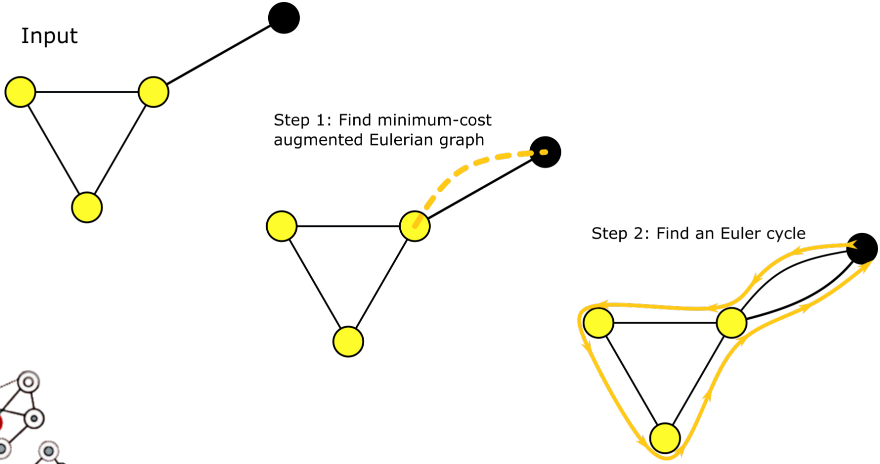
- edge demand ≥ 0



Complexity


CCPP
RPP
CARP
NP-Hard
NP-Complete
Not solvable in polynomial time
NP
Check solutions in polynomial time
CPP
Solvable in polynomial time
P
Difficult
Complexity
Easy
If $P \neq NP$

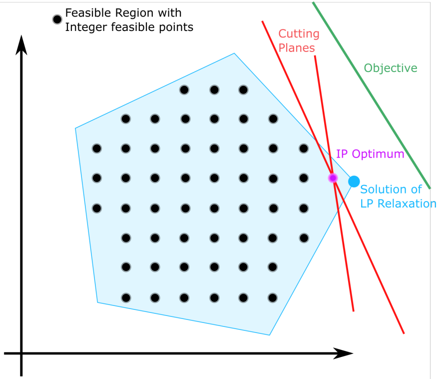
Exact Solutions to the CPP



Integer Linear Programming

Integer Linear Programming:
- optimize objective function
- requirements as linear relationships
- some variables restricted to be integer
- NP-complete

Branch and Cut Algorithm:
- solve without integer constraints
- cutting plane algorithm
- branch and bound into multiple sub-problems



Step 1: Find the minimum-cost augmentation:

Minimize

$$\sum_{(v_i,v_j)\in E} c_{ij}x_{ij} \tag{1}$$

subject to

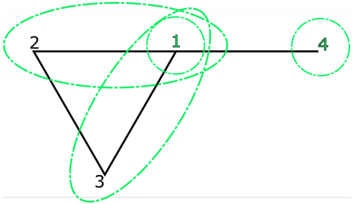
$$\sum_{(v_i,v_j)\in A(S)} x_{ij} \geq 1 \qquad (S \subset V, \text{S odd}) \tag{2}$$

$$x_{ij} \geq 0 \qquad ((v_i, v_j) \in E) \tag{3}$$

$$x_{ij} \text{ integer} \qquad ((v_i, v_j) \in E) \tag{4}$$

- x_{ij}**, the number of times the edge** (v_i, v_j) **is used in the solution**
  so the number of copies of each edge added to augment the graph
- $A(S)$**, all edges between the vertex set** $S \subset V$ **and its complement**
  $A(S) = \{(v_i, v_j); v_i \in S, v_j \in V \setminus S \text{ or } v_j \in S, v_i \in V \setminus S\}$

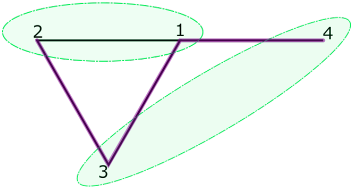
Some of the proper odd vertex subsets S:



Apply the constraint from equation (2) to all of them.
Here is an example choosing one possible subset:

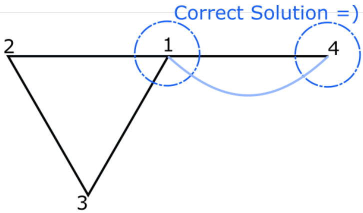
S = {2, 1}
A(S) = {(1,4); (1,3); (2,3)}

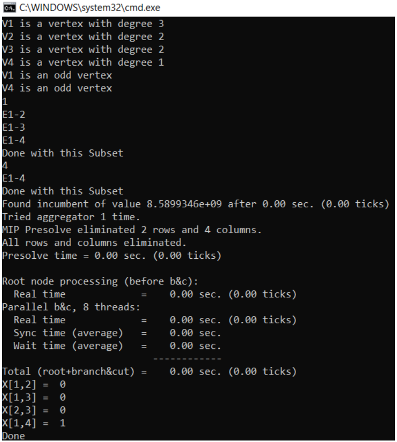
$$x_{13} + x_{14} + x_{23} \geq 1$$



Constraint $\sum_{(v_i,v_j)\in A(S)} x_{ij} \geq 1$ for

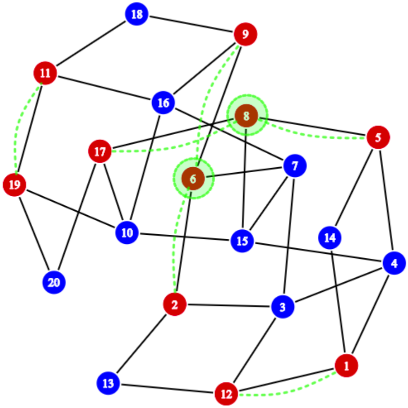
- all proper odd subsets S with 1 element
- works well for very small examples





```
C:\WINDOWS\system32\cmd.exe
V1 is a vertex with degree 3
V2 is a vertex with degree 2
V3 is a vertex with degree 2
V4 is a vertex with degree 1
V1 is an odd vertex
V4 is an odd vertex
1
E1-2
E1-3
E1-4
Done with this Subset
4
E1-4
Done with this Subset
Found incumbent of value 8.5899346e+09 after 0.00 sec. (0.00 ticks)
Tried aggregator 1 time.
MIP Presolve eliminated 2 rows and 4 columns.
All rows and columns eliminated.
Presolve time = 0.00 sec. (0.00 ticks)

Root node processing (before b&c):
  Real time             =    0.00 sec. (0.00 ticks)
Parallel b&c, 8 threads:
  Real time             =    0.00 sec. (0.00 ticks)
  Sync time (average)   =    0.00 sec.
  Wait time (average)   =    0.00 sec.
                          ------------
Total (root+branch&cut) =    0.00 sec. (0.00 ticks)
X[1,2] =  0
X[1,3] =  0
X[2,3] =  0
X[1,4] =  1
Done
```

But this algorithm doesn’t give an optimal solution for more complicated examples like this one.

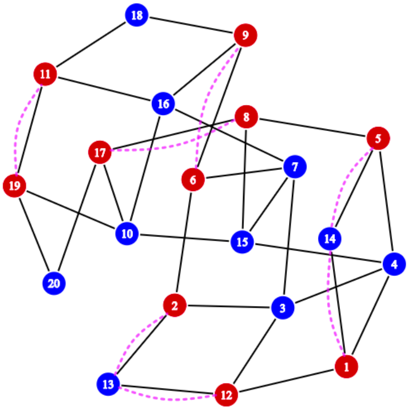


C:\WINDOWS\system32\cmd.exe

```
Found incumbent of value 6.6571993e+10 after 0.00 sec. (0.00 ticks)
Tried aggregator 1 time.
MIP Presolve eliminated 10 rows and 29 columns.
All rows and columns eliminated.
Presolve time = 0.02 sec. (0.02 ticks)

Root node processing (before b&c):
  Real time             =    0.02 sec. (0.02 ticks)
Parallel b&c, 8 threads:
  Real time             =    0.00 sec. (0.00 ticks)
  Sync time (average)   =    0.00 sec.
  Wait time (average)   =    0.00 sec.
                          ------------
Total (root+branch&cut) =    0.02 sec. (0.02 ticks)
X[1,12] =  1
X[1,14] =  0
X[12,13] =  0
X[13,2] =  0
X[3,12] =  0
X[1,4] =  0
X[5,14] =  0
X[2,3] =  0
X[3,4] =  0
X[5,4] =  0
X[5,8] =  1
X[4,15] =  0
X[3,7] =  0
X[6,2] =  1
X[6,7] =  0
X[7,15] =  0
X[8,15] =  0
X[8,17] =  1
X[10,15] =  0
X[7,16] =  0
X[6,9] =  1
X[9,16] =  0
X[10,16] =  0
X[10,17] =  0
X[17,20] =  0
X[10,19] =  0
X[11,16] =  0
X[9,18] =  0
X[11,18] =  0
X[11,19] =  1
X[19,20] =  0
Done
```

Step 1: Creating an algorithm to solve constraint (2) for

- all proper odd subsets with 1 element
- all possible combinations of *k* elements of the vertex set ($2 \leq k \leq n-2$) where the combination is an odd subset

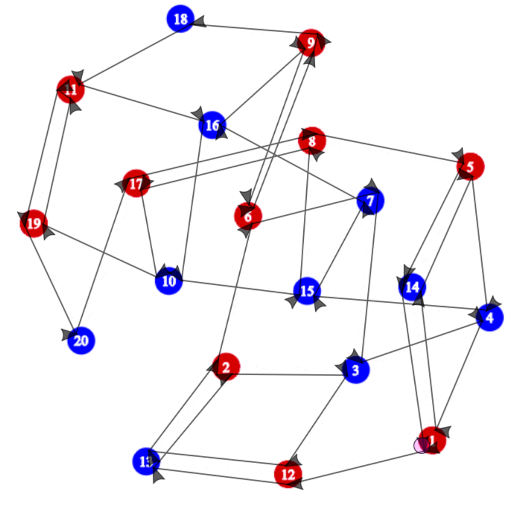


C:\WINDOWS\system32\cmd.exe

```
We creat a constraint for a set with vertices V12, V13, V2, V3, V4, V5, V8, V15,
Found incumbent of value 6.6571993e+10 after 0.02 sec. (28.25 ticks)
Tried aggregator 1 time.
MIP Presolve eliminated 262134 rows and 0 columns.
Reduced MIP has 262144 rows, 31 columns, and 4063232 nonzeros.
Reduced MIP has 31 binaries, 0 generals, 0 SOSs, and 0 indicators.
Presolve time = 2.03 sec. (1399.17 ticks)
Probing time = 0.13 sec. (29.17 ticks)
Tried aggregator 1 time.
Reduced MIP has 262144 rows, 31 columns, and 4063232 nonzeros.
Reduced MIP has 31 binaries, 0 generals, 0 SOSs, and 0 indicators.
Presolve time = 3.47 sec. (1916.18 ticks)
Probing time = 0.19 sec. (29.84 ticks)
MIP emphasis: balance optimality and feasibility.
MIP search method: dynamic search.
Parallel mode: deterministic, using up to 8 threads.
Root relaxation solution time = 3.69 sec. (1457.12 ticks)

        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

*     0+    0                      6.65720e+10        0.0000             100.00%
*     0+    0                           7.0000        0.0000             100.00%
      0     0        cutoff             7.0000        7.0000       34    0.00%
      0     0        cutoff             7.0000        7.0000       34    0.00%
Elapsed time = 11.14 sec. (6023.31 ticks, tree = 0.01 MB, solutions = 1)

Root node processing (before b&c):
  Real time             =   11.17 sec. (6023.31 ticks)
Parallel b&c, 8 threads:
  Real time             =    0.00 sec. (0.00 ticks)
  Sync time (average)   =    0.00 sec.
  Wait time (average)   =    0.00 sec.
                          ------------
Total (root+branch&cut) =   11.17 sec. (6023.31 ticks)
X[1,12] =  0
X[1,14] =  1
X[12,13] =  1
X[13,2] =  1
X[3,12] =  0
X[1,4] =  0
X[5,14] =  1
X[2,3] =  0
X[3,4] =  0
X[5,4] =  0
X[5,8] =  0
X[4,15] =  0
X[3,7] =  0
X[6,2] =  0
X[6,7] =  0
X[7,15] =  0
X[8,15] =  0
X[8,17] =  1
X[10,15] =  0
X[7,16] =  0
X[6,9] =  1
X[9,16] =  0
X[10,16] =  0
X[10,17] =  0
X[17,20] =  0
X[10,19] =  0
X[11,16] =  0
X[9,18] =  0
X[11,18] =  0
X[11,19] =  1
X[19,20] =  0
Done
```

Step 2: Find an Euler Cycle in the augmented graph using Fleury's Algorithm

1. Start at an arbitrary vertex v_i traverse an edge (v_i, v_j) that is not a bridge and erase edge (v_i, v_j)
2. Set $v_i := v_j$ and repeat step 1 starting from v_j or stop if all edges have been deleted.




```
The Euler Cycle includes:
E[1,12]
E[12,13]
E[13,2]
E[2,3]
E[3,12]
E[12,13]
E[13,2]
E[2,6]
E[6,7]
E[7,3]
E[3,4]
E[4,1]
E[1,14]
E[14,5]
E[5,4]
E[4,15]
E[15,7]
E[7,16]
E[16,9]
E[9,6]
E[6,9]
E[9,18]
E[18,11]
E[11,16]
E[16,10]
E[10,15]
E[15,8]
E[8,17]
E[17,10]
E[10,19]
E[19,11]
E[11,19]
E[19,20]
E[20,17]
E[17,8]
E[8,5]
E[5,14]
E[14,1]
```

Other Areas of Interest

- Consider other types arc routing problems
  - mixed or directed graphs
  - capacitated problems
  - extend constraints to practical problems
- Solving large-scale instances quickly and reliably
  - Trade-off between quality of solution and running time
  - Consider best ways to relax constraints
  - Find good heuristic algorithms and upper bounds
  - Find good lower bounds on the optimal solution

References

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