On the Classification of Motions of Laman Graphs

Georg Grasegger, Jan Legerský, Josef Schicho

RISC JKU Linz, Austria

Geometric constraint systems: rigidity, flexibility and applications
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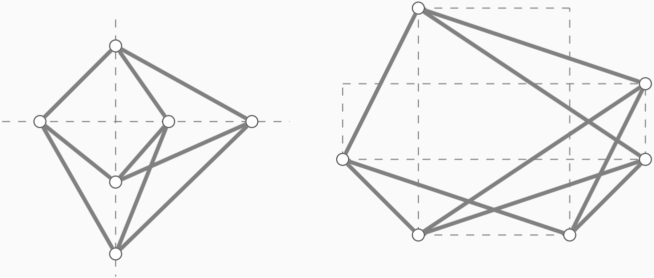
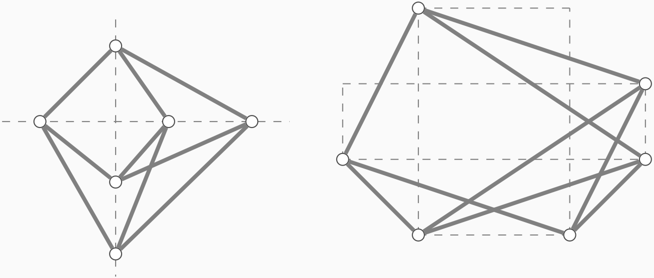


Dixon (1899), Walter and Husty (2007)



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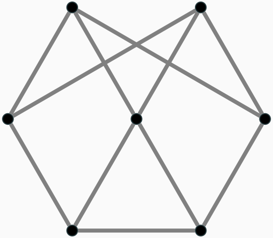


Q_1

Proper flexible labelings

An edge labeling $\lambda : E \to \mathbb{R}_+$ of a graph $G = (V, E)$ is called *flexible* if there are infinitely many non-congruent realizations $\rho : V \to \mathbb{R}^2$ such that $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges uv in E.

$$(x_{\bar{u}}, y_{\bar{u}}) = (0, 0)$$
$$(x_{\bar{v}}, y_{\bar{v}}) = (\lambda(\bar{u}\bar{v}), 0)$$
$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(uv)^2, \quad \forall\, uv \in E$$

An irreducible component of the solution set is called an *algebraic motion*.

Proper flexible labelings

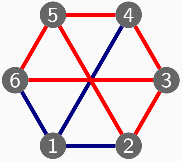
An edge labeling $\lambda : E \to \mathbb{R}_+$ of a graph $G = (V, E)$ is called *proper flexible* if there are infinitely many non-congruent injective realizations $\rho : V \to \mathbb{R}^2$ such that $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges uv in E.

$$
\begin{aligned}
(x_{\bar{u}}, y_{\bar{u}}) &= (0, 0) \\
(x_{\bar{v}}, y_{\bar{v}}) &= (\lambda(\bar{u}\bar{v}), 0) \\
(x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda(uv)^2, \quad \forall\, uv \in E \\
(x_u - x_v)^2 + (y_u - y_v)^2 &\neq 0, \quad \forall\, uv \notin E\,.
\end{aligned}
$$

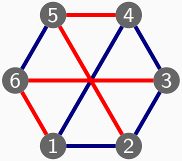
NAC-colorings

Definition

A coloring of edges $\delta : E \to \{\text{blue, red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.



ω_1



ϵ_{56}

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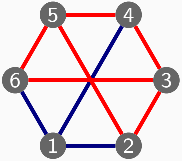

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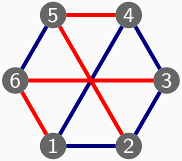

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Theorem (GLS)

A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.

Active NAC-colorings

$$\lambda_{uv}^2 = (x_v - x_u)^2 + (y_v - y_u)^2$$
$$= \underbrace{((x_v - x_u) + i(y_v - y_u))}_{W_{u,v}} \underbrace{((x_v - x_u) - i(y_v - y_u))}_{Z_{u,v}}$$

Lemma (GLS)

Let λ be a flexible labeling of a graph G. Let $\mathcal{C}$ be an algebraic motion of (G, λ). If $\alpha \in \mathbb{Q}$ and ν is a valuation of the complex function field of $\mathcal{C}$ such that there exists edges $\bar{u}\bar{v}, \hat{u}\hat{v}$ with $\nu(W_{\bar{u},\bar{v}}) = \alpha$ and $\nu(W_{\hat{u},\hat{v}}) > \alpha$, then $\delta : E_G \to \{red, blue\}$ given by

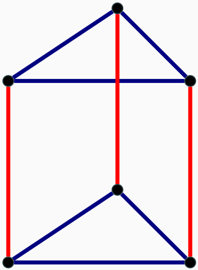
$$\delta(uv) = red \iff \nu(W_{u,v}) > \alpha\,,$$
$$\delta(uv) = blue \iff \nu(W_{u,v}) \leq \alpha\,.$$

is a NAC-coloring, called active.

Active NAC-colorings of quadrilaterals

Quadrilateral	Motion	Active NAC-colorings
Rhombus	parallel	
	degenerate	 resp. 
Parallelogram		
Antiparallelogram		
Deltoid	nondegenerate	
	degenerate	
General		

Three-prism



Leading coefficient system

Assume a valuation that gives only one active NAC-coloring
$\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n)$:

$$\sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{red}}} \underbrace{(w_{u_i u_{i+1}} t + \text{h.o.t.})}_{W_{u_i, u_{i+1}}} + \sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{blue}}} \underbrace{(w_{u_i u_{i+1}} + \text{h.o.t.})}_{W_{u_i, u_{i+1}}} = 0\,.$$

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$$\sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{blue}}} w_{u_i u_{i+1}} = 0\,.$$

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$$\sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{blue}}} w_{u_i u_{i+1}} = 0\,.$$

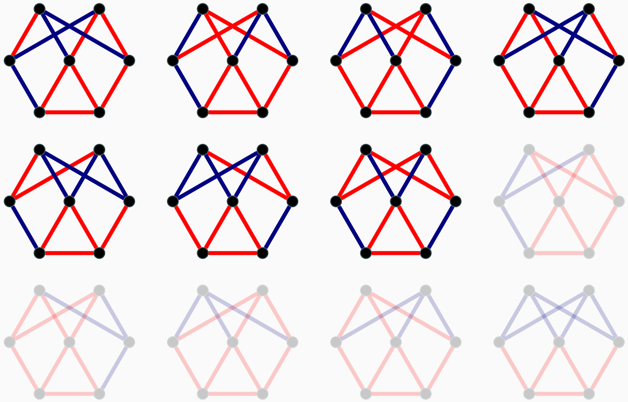
For all $uv \in E_G$:

$$w_{uv} z_{uv} = \lambda_{uv}^2\,.$$

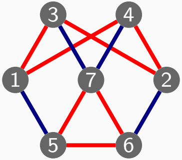
$\implies$ elimination using Gröbner basis provides an equation in λ_{uv}'s.

Singleton NAC-colorings

If a valuation yields two active NAC-colorings δ, δ', then the set $\{(\delta(e), \delta'(e)) \colon e \in E_G\}$ has 3 elements.



Triangle in $Q1$



$$\implies \lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right) rs = 0\,,$$

$$r = \lambda_{24}^2 - \lambda_{23}^2,\; s = \lambda_{14}^2 - \lambda_{13}^2$$

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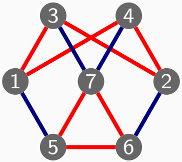

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Considering the equation as a polynomial in r, the discriminant is

$$(\lambda_{56}+\lambda_{57}+\lambda_{67})(\lambda_{56}+\lambda_{57}-\lambda_{67})(\lambda_{56}-\lambda_{57}+\lambda_{67})(\lambda_{56}-\lambda_{57}-\lambda_{67})s^2\,.$$

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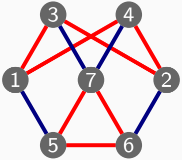

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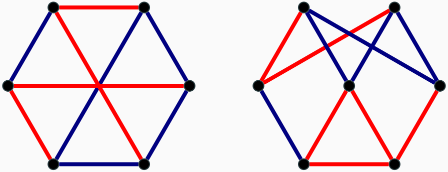
Theorem (GLS)

The vertices 5, 6 and 7 are collinear for every proper flexible labeling of Q_1.

Orthogonal diagonals



Lemma (GLS)

If there is an active NAC-coloring δ of an algebraic motion of (G, λ) such that a 4-cycle $(1, 2, 3, 4)$ is blue and there are red paths from 1 to 3 and from 2 to 4, then

$$\lambda_{12}^2 + \lambda_{34}^2 = \lambda_{23}^2 + \lambda_{14}^2 \,,$$

namely, the 4-cycle $(1, 2, 3, 4)$ has orthogonal diagonals.

Ramification formula

Theorem (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G, λ) with the set of active NAC-colorings N. There exists $\mu_\delta \in \mathbb{Z}_{\geq 0}$ for all NAC-colorings δ of G such that:

1. $\mu_\delta \neq 0$ *if and only if* $\delta \in N$, *and*
2. *for every 4-cycle* (V_i, E_i) *of* G, *there exists a positive integer* d_i *such that*

$$\sum_{\substack{\delta \in \mathrm{NAC}_G \\ \delta|_{E_i} = \delta'}} \mu_\delta = d_i \qquad \text{for all } \delta' \in \{\delta|_{E_i} : \delta \in N\}.$$

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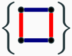
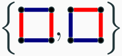
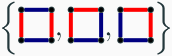
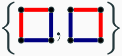
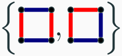
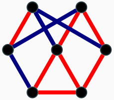
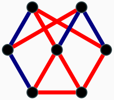
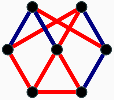
$\mathfrak{p} =$  , $\mathfrak{o} =$  , $\mathfrak{g} =$  ,

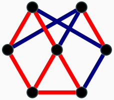
$\mathfrak{a} =$  , $\mathfrak{e} =$  .

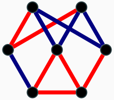
Example

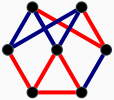

ϵ_{13}

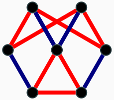

ϵ_{14}

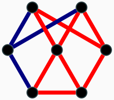

ϵ_{23}

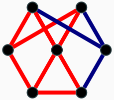

ϵ_{24}

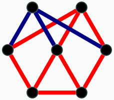

γ_1

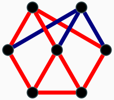

γ_2

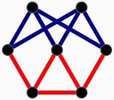

η


ψ_1

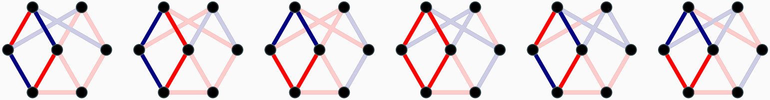

ψ_2


ϕ_3

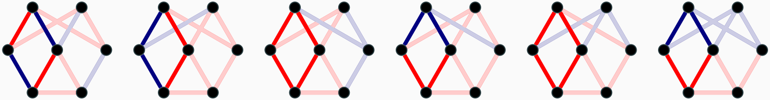

ϕ_4


ζ

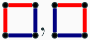
Example



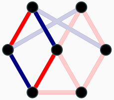
ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2

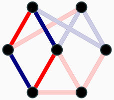


η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

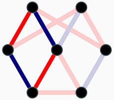
Antiparallelogram () $\Longrightarrow$

Example

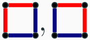




ϵ_{13}	ϵ_{14}	ϵ_{23}	ϵ_{24}	γ_1	γ_2

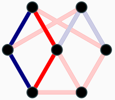


η	ψ_1	ψ_2	ϕ_3	ϕ_4	ζ

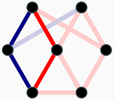
Antiparallelogram () $\implies$

$$\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0$$

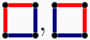
Example



ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2



η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

Antiparallelogram () $\Longrightarrow$

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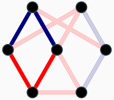
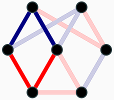
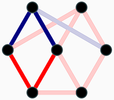
$$\mu_{\epsilon_{14}} + \mu_{\psi_1}$$

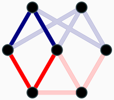
Example



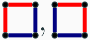


ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2





η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

Antiparallelogram () $\Longrightarrow$

$$\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0$$

$$\mu_{\epsilon_{14}} + \mu_{\psi_1} = \mu_{\epsilon_{23}} + \mu_{\gamma_2} + \mu_{\phi_3} + \mu_\zeta$$

Classification of motions

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- Remove combinations with coinciding vertices (due to edge lengths, perpendicular diagonals)
- Identify symmetric cases
- Compute necessary conditions for λ_{uv}'s using leading coefficient systems
- Check if there is a proper flexible labeling satisfying the necessary conditions

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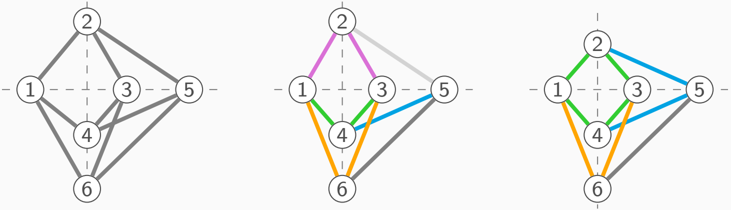
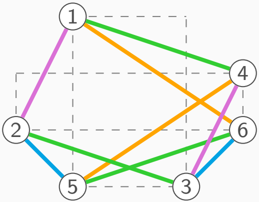
- Remove combinations with coinciding vertices (due to edge lengths, perpendicular diagonals)
- Identify symmetric cases
- Compute necessary conditions for λ_{uv}'s using leading coefficient systems
- Check if there is a proper flexible labeling satisfying the necessary conditions

Implementation – SageMath package `FlexRiLoG`
(`https://github.com/Legersky/flexrilog`)



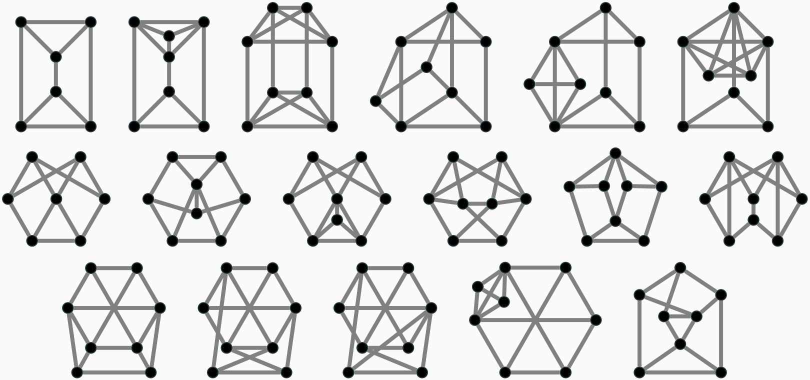
4-cycles	active NAC-colorings	#	
$\mathfrak{ggggggggg}$	$\mathrm{NAC}_{K_{3,3}}$	1	
$\mathfrak{ooogggggg}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3\}$	6	Dixon I
$\mathfrak{pooggogge}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}\}$	9	
$\mathfrak{pgggaggag}$	$\{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6\}$	18	Dixon II



Classification of motions of Q_1

4-cycles	active NAC-colorings	#	type	dim.
$\mathfrak{pggpgpg}$	$\{\epsilon_{13}, \epsilon_{24}, \eta\}$	2	I	4
$\mathfrak{poapope}$	$\{\epsilon_{13}, \eta\}$	4	$\subset$ I, IV$_-$, V, VI	2
$\mathfrak{peepapa}$	$\{\epsilon_{13}, \epsilon_{24}\}$	2	$\subset$ I, II, III	2
$\mathfrak{ogggggg}$	$\{\epsilon_{ij}, \gamma_1, \gamma_2, \psi_1, \psi_2\}$	1	II$_- \cup$ II$_+$	5
$\mathfrak{peegggg}$	$\{\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}\}$	1	$\subset$ II$_-$, II$_+$	4
$\mathfrak{oggpgga}$	$\{\epsilon_{13}, \epsilon_{24}, \gamma_1, \psi_2\}$	4	$\subset$ II$_-$	3
$\mathfrak{oggegge}$	$\{\epsilon_{13}, \epsilon_{23}, \gamma_1, \gamma_2\}$	2	$\subset$ II$_-$, deg.	2
$\mathfrak{ogggaga}$	$\{\epsilon_{13}, \epsilon_{24}, \psi_1, \psi_2, \zeta\}$	2	III	3
$\mathfrak{ggapggg}$	$\{\epsilon_{13}, \eta, \phi_4, \psi_2\}$	4	IV$_- \cup$ IV$_+$	4
$\mathfrak{ggaegpe}$	$\{\epsilon_{13}, \eta, \gamma_2, \phi_3\}$	4	V	3
$\mathfrak{pggegge}$	$\{\epsilon_{13}, \epsilon_{23}, \eta, \zeta\}$	2	VI	3

Other graphs



Thank you

jan.legersky@risc.jku.at
jan.legersky.cz

Animations