Lancaster University
Management School

Economics Working Paper Series

2013/003

Higher-order Moments in the Theory of Diversification and Portfolio Composition

Trino-Manuel Ñíguez, Ivan Paya, David Peel and Javier Perote

The Department of Economics
Lancaster University Management School
Lancaster LA1 4YX
UK

LUMS home page: http://www.lums.lancs.ac.uk/

Higher-order moments in the theory of diversification and portfolio composition

Trino-Manuel Ñíguez

Department of Economics and Quantitative Methods, Westminster Business School, University of Westminster, London NW1 5LS, UK

Ivan Paya

Department of Economics, Lancaster University Management School, Lancaster LA1 4YX, UK

David Peel

Department of Economics, Lancaster University Management School, Lancaster LA1 4YX, UK

Javier Perote

Department of Economics, University of Salamanca, 37071 Salamanca, Spain

Abstract

This paper examines the role of higher-order moments in portfolio choice within an expected-utility framework. We consider two-, three-, four- and five-parameter density functions for portfolio returns and derive exact conditions under which investors would all be optimally *plungers* rather than *diversifiers*. Through comparative statics we show the importance of higher-order risk preference properties, such as *riskiness*, *prudence* and *temperance*, in determining *plunging* behaviour. Empirical estimates for the S&P500 provide evidence for the optimality of diversification.

JEL classification: C14, C22, G11.

Keywords: Generalized beta distribution; Higher-order moments; Portfolio choice; Prudence; Semi-nonparametric distributions; Temperance.

1 Introduction

Feldstein (1969) in a classic paper on optimal allocation of wealth between risk free and risky assets, demonstrated that under log-utility and log-normality, the investor's decision to *plunge*, i.e., allocating all wealth in the risky asset, could occur under reasonable values of the mean and variance of the portfolio return. This analysis was a counter example to the result of Tobin (1958) on the sufficiency of risk aversion (quadratic utility) under two-parameter distributions to ensure diversification. Generalizing this analysis, Meyer (1987) showed that these results are valid for all classes of two-parameter distributions with mean and variance equivalent to measures of location and scale, irrespective of the utility function. More recently, Boyle and Conniffe (2008) examine the equivalence of expected utility (EU) and mean-variance (MV) approaches for non location-scale distributions. Although, Tobin's and Feldstein's seminal results on *plunging* were extensively discussed and treated in the literature on portfolio theory,[1] a central aspect remains not satisfactorily addressed.[2] Namely, since the share of wealth allocated to the risky asset obtained from Feldstein's (1969) EU model is optimally determined, why do we not observe *plunging* in practice?[3]

In this paper we revisit this issue focusing on the effect of higher-order moments.[4] It is now commonly accepted that those higher-moments do affect investor's decisions. However, we find in the literature different theoretical arguments that support that effect. Menezes et al. (1980) develop the concept of downside risk (DR hereafter) within a choice-theoretic framework and provide a relationship between the third derivative of the utility function and individuals' risk preferences. Their definition allows the distinction between increasing DR and riskiness because probability distribution functions (pdfs hereafter) that can be obtained as mean-variance-preserving transformations of other pdfs will exhibit more DR. Pdfs are therefore either comparable in terms of riskiness or DR but not in terms of both. A

[1]See, for instance, Borch (1969), Tobin (1969), Glustoff and Nigro (1972), Mayshar (1978), Feldstein (1978) and Goldman (1979), among others

[2]Within the dual theory of choice, Yaari (1987) notes that preferences display *plunging* behavior. Chambers and Quiggin (2007) show that this is characteristic of the entire class of constant risk-averse, quasi-concave preferences.

[3]See Haliassos and Betaut (1995) for evidence on the optimal decision of investors' liquidity in a non-EU framework.

[4]The issue of a corner solution has recently been explicitly discussed within a MV model in Ormiston and Schlee (2001). They discussed the comparative statics of EU versus MV, and provided necessary and sufficient condition for an interior solution (no-*plunging*), acknowledging the limitations of MV analysis with regards to higher-order moments. Following these results, the MV model has been extended to include skewness; see Chunhachinda *et al.* (1997), Prakash *et al.* (2003) and Eichner and Wagener (2010).

distribution function that has less DR than another will also be more right skewed, although the converse is not necessarily true.

An equivalent concept to DR, i.e. "prudence", has been defined using agents' optimizing behavior. The importance of the third derivative of utility in determining demand for precautionary savings defines prudence according to Kimball (1990). Behavioral aspects of investors have also been related to the fourth derivative of the utility function, "temperance" (see Kimball, 1992), or the fifth derivative, "edginess" (see Lajeri-Chaherli, 2004). More recently, Eeckhoudt and Schlesinger (2006) define all those risk preference properties, and others of higher order, i.e. "risk apportionment of order n", by preferences toward particular classes of lotteries, and show that they are equivalent to signing the n^{th} derivative of the utility function within an EU framework. It is therefore the case that prudence (or DR), temperance, and edginess are "pure" third-, fourth- and fifth-order effects, respectively, whilst decreasing absolute risk aversion (DARA), "properness", risk vulnerability or standard risk aversion include effects of other orders. These pure n^{th} order effects can be related to stochastic dominance of order n (SD_n) even though they are not equivalent concepts, since utility functions that define SD_n are a subset of the ones that define SD_{n-1}.

An alternative approach is based on the relation between individual preferences for risk and moments of the distribution, through utility approximations. Levy (1969) extended the EU model in Tobin (1958) and Feldstein (1969) using the classical MV framework of Markowitz (1952) (see also Adler, 1969 and Miller, 1975) to show that for linear utility functions of order n, only the first n moments matter for the investor's liquidity decision, irrespective of the number of parameters of the pdf.[5] In particular, Horvath and Scott (1985) show, using a cubic utility function, that an EU maximizer investor is more likely to change drastically the composition of the portfolio towards the riskier asset when the skewness of the distribution of returns consistently increases relative to the variance. Jurczenko and Maillet (2006) presented the theoretical framework of utility specifications and multi-moment decision criteria in an EU model, and developed a quadratic utility specification to derive an exact decision criterion in terms of the first four moments. They determined the preference and distributional restrictions needed to ensure that utility approximations, written in terms of moments, do preserve the individual preference ranking.[6]

In order to take all those theoretical developments into consideration we take into account

[5]Further discussion on the specific role of skewness on portfolio choice can be seen in Arditti (1967), Arditti and Levy (1975), Kraus and Litzenberger (1976), Simkowitz and Beedles (1978) and Kane (1982).

[6]Empirical studies on the effect of higher-order moments in EU models can be found in Brandt *et al.* (2005), and Jondeau and Rockinger (2006).

the progress made to capture, with different degrees of accuracy, the stylized features of asset returns (Mandelbrot 1963, Fama 1965). In particular, we examine the effect of higher-order moments on portfolio choice through parametric and semi-nonparametric (SNP) pdfs widely used in the literature to model asset returns asymmetric and leptokurtic distribution. First, we consider the five-parameter weighted generalized beta distribution of the second kind (WGB2) and the four-parameter generalized beta type 2 (GB2) pdf, which nests the generalized gamma (GG) which, in turns, nests the log-normal, gamma, Weibull and many other distributions (see McDonald 1984, Bookstaber and McDonald 1987, Mittnik and Rachev 1993, McDonald and Xu 1995, Jensen *et al.* 2003, and Ye *et al.*, 2012, for the theoretical properties of these densities and applications to economic data). Second, we consider the case of returns distributed according to a logarithmic semi non-parametric (log-SNP) pdf. Log-SNP pdfs encompass the log-normal and are characterized by its flexibility to fit any empirical distribution to any degree of accuracy depending on the density function truncation order (see Corrado and Su, 1996, Jondeau and Rockinger, 2001, Ñíguez *et al.*, 2012, and Ñíguez and Perote, 2012, for applications of (log)-SNP densities in economics and finance).

The contribution of the paper is to formally derive the conditions that show how the higher-order moments of the pdfs affect the investor's decision to diversify and whether those conditions are related to different attitudes toward risk, such as prudence and temperance, in our simple, but theoretically important, model structure.[7] The conditions derived theoretically do not find support in empirical estimates for the S&P500 implying that investors' optimal choice would be to diversify.

The structure of the paper is as follows. In section 2 we analyze portfolio choice decisions under log utility and parametric and semi-parametric distributions for wealth returns: the WGB2 and its special cases (GB2, GG, gamma , Weibull and log-normal), and the log-SNP. In Section 3 we provide an application of our analysis for the S&P500. The final section is a brief conclusion.

[7] Boyle and Conniffe (2005) discussed alternative two-parameter pdfs together with different utility functions and showed that the likelihood of a risky-asset-only portfolio is higher with some distributions than others, whilst the core of this paper presents exact *plunging* conditions, providing a formal approach.

2 Plunging with log utility under alternative distributions

Following Tobin (1958) let us consider a two-asset (risky/riskless) economy in which an investor with initial wealth $\overline{\omega}_0$ decides to invest a proportion, $0 \leq \theta \leq 1$, in the risky asset so that after one period expected wealth becomes

$$\overline{\omega} = (1-\theta)\overline{\omega}_0 + \theta\overline{\omega}_0 E(r), \tag{1}$$

where $E(r)$ is the expected gross rate of return of the risky asset ($r \geq 1$).[8] Expected wealth risk is traditionally measured by the standard deviation, σ, assuming normality on the pdf of r, hereafter denoted as f. We argue that the assumption of normality here may lead to a significant bias in the model outcome, i.e. the optimal demand for liquidity, as it is a well-known fact that r is not normally distributed but its pdf is significantly skewed and leptokurtic (see e.g. Mandelbrot, 1963). Thus, we relax this assumption and study the effect of alternative pdfs in the model, focusing on explaining the controversial corner solution (*plunging,* $\theta = 1$).

For the investor's preferences on portfolio choices (θ) we assume a typical log-utility function, $u_1(\omega) = \ln(\omega)$, which presents decreasing absolute risk aversion (DARA), and constant relative risk aversion (CRRA) of 1.[9] In Appendix A we provide an extension to the discussion in this section by considering an alternative (power) utility function, which can exhibit smaller degree of relative risk aversion. These two utility functions display the less restrictive features that characterize prudence (or DR) and temperance, that is $u''' > 0$, and $u^{iv} < 0$, respectively (see Eeckhoudt and Schlesinger, 2006).

Introducing the core notation of the paper: Consider an investor who maximizes her EU by choosing the proportion θ to invest in the risky asset, so her objective program is (2),

$$\begin{aligned} \max_{\{\theta\}} E_f\left[u(\omega)\right] &= \max_{\{\theta\}} E_f\left[u\left((1-\theta)\overline{\omega}_0 + \theta\overline{\omega}_0 r\right)\right] \\ &= \max_{\{\theta\}} E_f\left\{u\left(\overline{\omega}_0\left[1+\theta(r-1)\right]\right)\right\}, \end{aligned} \tag{2}$$

For the sake of simplicity, let denote $\xi_f(u(\omega);\theta) = \frac{\partial E_f[u(\omega)]}{\partial\theta}$. Therefore, $\theta = 1$ is the solution

[8] Therefore, the investor's strategy of short selling is ruled out.

[9] Amongst others, we note that the empirical evidence reported by Chetty (2006) and Bombardini and Trebbi (2012), in the context of labour supply and attiudes to risk in a game show, respectively, suggests that log utility may be a good approximation to agents utility function.

to (2) if both conditions (3) and (4) hold,

$$\xi_f(u(\omega);\theta) > 0 \;\forall\theta \in [0,1) \tag{3}$$

$$\xi_f(u(\omega);\theta)\big|_{\theta=1} \geq 0. \tag{4}$$

Besides, if $\xi_f(u(\omega);\theta)\big|_{\theta=0}$ is positive and strictly decreasing with θ, i.e. $\frac{\partial^2 E_f[u(\omega)]}{\partial\theta^2} < 0$, so $E_f[u(\omega)]$ is a strictly increasing and concave function of θ, then *plunging* is optimal if $\xi_f(u(\omega);\theta)\big|_{\theta=1} \geq 0$; see Feldstein (1969).

Thus, it is clear that the existence of a corner solution in this EU framework depends on both the investor's utility function and the risky asset return pdf. In particular, for a log-utility function the maximization program (2) becomes

$$\max_{\{\theta\}} \left(E_f\left[u_1(\omega)\right]\right) = \max_{\{\theta\}} \left(\ln(\overline{\omega}_0) + E_f\left\{\ln\left[1+\theta\left(r-1\right)\right]\right\}\right), \tag{5}$$

thus the conditions for a corner solution are given by

$$\xi_f(u_1(\omega);\theta) = E_f\left[\frac{r-1}{1+\theta\left(r-1\right)}\right] > 0 \;\forall\theta \in [0,1) \tag{6}$$

$$\xi_f(u_1(\omega);\theta)\big|_{\theta=1} = 1 - E_f\left(r^{-1}\right) \geq 0. \tag{7}$$

Provided that $E_f(r) > 1$, the function $\xi_f(u_1(\omega);\theta)$ (equation (6)) is positive and decreasing with $\theta \in [0,1)$, therefore $\theta = 1$ is optimal if $\xi_f(u_1(\omega);\theta)\big|_{\theta=1} \geq 0$.

2.1 Return distributions: generalized beta type 2

Table 1 displays the density and moment generating functions (mgf) for the five-, four- and three-parameter generalized distributions we consider in the paper, WGB2, GB2, and GG, respectively. The latter two distributions have a longer tradition and have already been employed to fit the distribution of asset returns (see Bookstaber and McDonald 1987, Mittnik and Rachev 1993, McDonald and Xu 1995, and Jensen *et al.* 2003).

[Insert Table 1 here]

McDonald (1984) demonstrates that the substitution $b = \frac{q^{\frac{1}{c}}}{a}$ as $q \to \infty$ in the GB2 density function generates the GG distribution[10] with shape parameters $a > 0$ and $c > 0$, and scale

[10]Note that the GG family nests many other distributions as special cases. For instance, gamma ($c = 1$), exponential $(p, c) = (1, 1)$, Weibull ($p = 1$), lognormal ($p \to \infty$), and Rayleigh $(p, c) = (1, 2)$.

parameter $p > 0$.[11] Thus, condition (7) for the GG is given by

$$\begin{aligned} 1 - E_f\left(r^{-1}\right) &= 1 - a\frac{\Gamma(p-\frac{1}{c})}{\Gamma(p)} \geq 0 \\ \mu_{1,GG} &\geq \frac{\Gamma(p+\frac{1}{c})}{\Gamma(p)}\frac{\Gamma(p-\frac{1}{c})}{\Gamma(p)}. \end{aligned} \tag{8}$$

This expression allows us to obtain results for other distributions nested within the GG. Table 2 below summarizes all these results and examples presented in this section.

[Insert Table 2 here]

Let us first consider the log-normal distribution following the classical literature on this topic. Using the pdf and mgf of the log-normal, the condition for optimum $\theta = 1$ is the following (see Appendix B),

$$m_{1,LN} \geq 1 + \frac{m_{2,LN}}{m_{1,LN}^2}. \tag{9}$$

Hereafter we will use the example in Feldstein (1969) (S&P500 returns) as a baseline for the comparative analysis on plunging behavior of the models we consider. Thus, assume that $m_{1,LN} = 1.05$, ($m_{t,f}$ denotes the *th*-central moment of pdf f), then investors would *plunge* under log-normality if $m_{2,LN} \leq 0.055125$, or similarly, unless the standard deviation is more than four times the expected net return, i.e., $m_{2,LN}^{1/2} > 0.23479$.[12] This threshold value is not unreasonable, hence the question of why we do not seem to observe more investors behaving as *plungers*.

We now examine how an alternative two-parameter pdf yields a different lower bound for the risky-asset-only portfolio for which we provide two examples. First, for the gamma distribution ($c = 1$ in expression for GG, Table 1), condition (8) is obtained as

$$1 - E_f\left(r^{-1}\right) = 1 - a\frac{\Gamma(p-\frac{1}{1})}{\Gamma(p)} \geq 0, \tag{10}$$

which can be expressed in terms of central moments as,

$$\begin{aligned} 1 - \frac{a}{p-1} &= 1 - \frac{\frac{m_{2,g}}{m_{1,g}}}{\frac{m_{1,g}^2}{m_{2,g}} - 1} \geq 0, \\ m_{1,g} &\geq 1 + \frac{m_{2,g}}{m_{1,g}}. \end{aligned} \tag{11}$$

[11] We do not consider values of $c < 1$ as they generate non-economically relevant distributions in some low value cases and in others do not change the results.

[12] In this case, the third central moment and standardized skewness (sk) of the distribution with $(m_{1,LN}, m_{2,LN}) = (1.05, 0.055125)$ are: $(m_{3,LN}, sk_{LN}) = (0.008826, 0.68198)$.

Following the baseline example, set $m_{1,g} = 1.05$, then *plunging* would occur with the gamma distribution if $m_{2,g} \leq 0.0525$. The third central moment and the standardized skewness ($sk = m_3/m_2^{1/2}$) of the distribution corresponding to $(m_{1,g}, m_{2,g}) = (1.05, 0.0525)$ are: $(m_{3,g}, sk_g) = (0.00525, 0.43644)$. It is important to note that for $m_{1,g} = 1.05$ the values $(m_{2,g}, m_{3,g})$ are both smaller than those of the log-normal case.[13]

Second, for the Weibull distribution ($p = 1$ in expression for GG, Table 1), the corner solution holds when parameter $c \geq 5.83493$ assuming $m_{1,W} = 1.05$. When $(c, m_{1,W}) = (5.83493, 1.05)$, the second and third central moments are $(m_{2,W}, m_{3,W}) = (0.04358, -0.0032383)$, or similarly, $(m_{2,W}^{1/2}, sk_W) = (0.20876, -0.3559)$. In this case, *plunging* can occur when the skewness is negative. It is worth noting that the variance decreases as parameter c is increased for a given mean so that $m_{2,W} = 0.04358$ is the highest variance for which a risky-asset-only portfolio can occur.[14] As in the case of the gamma distribution we also find that for $m_{1,W} = 1.05$ the values $(m_{2,W}, m_{3,W})$ are both smaller than those of the log-normal case. Figure 1 illustrates the differences in the tails and peaks of the two-parameter pdfs considered here, namely, the log-normal, gamma, and Weibull distribution with the same mean and variance $(m_{1,f}, m_{2,f}) = (1.05, 0.055125)$.

[Insert Figure 1 here]

The implication that follows from the analysis of the GG with a mean of 1.05 is that investors become *plungers* if the variance is less than 0.055125 depending on the particular distribution considered and the precise number for skewness (see Table 2, Panel A).[15]

A point that illustrates the fact that higher-order moments matter for the investors' decision on portfolio composition is that we find other parameter values for the GG distribution that yield the same mean and variance as the log-normal but investors do diversify (see Table 2, Panel B). The expected utility for the two GG distributions is lower than for the log-normal, and this difference in the investors' portfolio allocation decision, conditional on having the same first two moments, is due to downside risk aversion or,

[13] Throughout the paper, for the sake of easing the replication of our results, we present parameter and moments' values with different decimal points as results depend crucially on the rounding.

[14] Consideration of other distributions nested in the GG shows that for some of them *plunging* cannot occur with a mean of 1.05 for the risky asset. These include the Chi-Squared (χ^2) $(m_{1,\chi^2}, m_{2,\chi^2}, m_{3,\chi^2}) = (1.05, 0.9975, 4.10025)$.

[15] This result could also be related to the concept of 'greater central riskiness' (GCR), see Gollier (1995). Gollier showed that a risk-averse EU maximiser increases her investment in the risky asset when the return distribution F is replaced by G if and only if there exists a real number m such that $\int_{-\infty}^{x} r dG(r) \geq m \int_{-\infty}^{x} r dF(r)$ for all $x \in R$.

equivalently, prudence, rather than riskiness (see Menezes et al., 1980; and Eeckhoudt and Schlesinger, 2006).[16] The two specific GG pdfs above imply more DR than the log-normal distribution, that is, they involve the transfer of risk leftward in a distribution, making the individual worse off by such a change and willing to diversify.

We now turn into the more flexible four-parameter GB2 distribution. For this case, using the mgf shown in Table 1 and the expression for b derived from the first raw moment $\mu_{1,GB2} = \frac{b\Gamma\left(p+\frac{1}{c}\right)\Gamma\left(q-\frac{1}{c}\right)}{\Gamma(p)\Gamma(q)}$, we can express condition (7) as,

$$\mu_{1,GB2} \geq \frac{\Gamma\left(p+\frac{1}{c}\right)\Gamma\left(q-\frac{1}{c}\right)\Gamma\left(p-\frac{1}{c}\right)\Gamma\left(q+\frac{1}{c}\right)}{\Gamma(p)^2\,\Gamma(q)^2}. \tag{12}$$

It is important to note that the specification of the density function is relevant to derive the conditions for plunging even within a class of distributions. We make the point that not only higher-order moments matter but the precise specification of the distribution function as well. In other words, the conditions for plunging for a nested specification may differ from those of the corresponding density within the general form. We illustrate this result with two examples of the GB2 distribution that nest either the Weibull and the gamma distributions and admit slightly higher variance (and skewness) for which $\theta = 1$ is optimum (see Table 2, Panel C).

A point worth making is that the distribution which is most conducive to *plunging* in the class defined by the GB2 for a mean of 1.05 is $(p, c, q, b) = (21.3, 2, 7.3, 0.5859)$ with $(m_{1,GB2}, m_{2,GB2}, m_{3,GB2}, sk_{GB2}) = (1.05, 0.0581, 0.0141, 1.0068)$. The use of a GB2 therefore increases the chances of corner solution in the sense that a higher variance is traded for higher skewness for that condition to hold. Figure 2 plots the two-parameter (log-normal) and four-parameter (GB2) distributions which are most conducive of risky-asset-only portfolio with $(m_{1,f}, m_{2,f}) = (1.05, 0.055125)$. We observe their differences in terms of asymmetry and heavy-tails for the same mean and variance.

[Insert Figure 2 here]

It is also worth noting that when the mean and standardized skewness of two distributions are the same, the agent can *plunge* with the distribution with higher variance but diversify in the one with the lower variance; this is due to a higher third central moment, m_3, in the former. An example is shown in Table 2, Panel D, where the expected utility of the distribution with lower variance is actually lower for the same mean. Consequently, the

[16] In particular, $EU_{LN} = 0.0243951$, (hereafter EU_f denotes EU under density f) and for the two GG densities in Table 2 Panel B, $EU_{GG} = 0.0227413$, and $EU_{GG} = 0.0237374$.

GB2 distribution appears to admit cases for which an agent's risky choices do not meet the definitional requirements of skew affine (see Eichner and Wagener, 2011).

We complete the analysis of the parametric pdfs with the recently developed five-parameter WGB2. We employ this density together with its corresponding condition for the corner solution of the portfolio problem, (13), to show that two distributions with the same first three moments could still imply different behavior in terms of portfolio diversification.

$$\mu_{1,WGB2} \geq \frac{\Gamma\left(p+\frac{k}{c}+\frac{1}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{1}{c}\right)\Gamma\left(p+\frac{k}{c}-\frac{1}{c}\right)\Gamma\left(q-\frac{k}{c}+\frac{1}{c}\right)}{\Gamma^2\left(p+\frac{k}{c}\right)\Gamma^2\left(q-\frac{k}{c}\right)} \tag{13}$$

The WGB2 density function with parameter values $(p, c, q, b, k) = (4.92879, 2.80226,\ 6.5,\ 1.1025791,\ 0.7)$ does share the same first three moments as the log-normal distribution in Table 2 Panel A, but the corner solution does not hold. The fourth central moment is higher for the WGB2 while its expected utility is lower.[17] In this case, the difference in the investor's decision within this EU framework is related to a fourth-order effect, or temperance (see Eeckhoudt and Schlesinger, 2006).

2.2 Return distribution: log-SNP

SNP densities are based on Edgeworth (1896, 1907) and Type A Gram (1883)-Charlier (1906) series (see also Chebyshev 1890), Sargan (1975) brought them into SNP econometrics. These density functions are mainly characterized by their flexibility to approximate the shape of any distribution of probabilities. During the last decades, SNP pdfs have been extensively developed by authors such as Jarrow and Rudd (1982), Gallant and Nychka (1987) and Jondeau and Rockinger (2001); recent theoretical results and applications in economic and financial modelling and forecasting are provided in León *et al.* (2009), Del Brio *et al.* (2011), and Ñíguez *et al.* (2012).

Appendix C contains the definition of the log-SNP pdf, its mgf, and a discussion of its properties. Under the log-SNP assumption the corner solution condition is given by

$$\xi_{\Upsilon_n}(u_1(\omega); \theta; m, \nu, \boldsymbol{\delta})\big|_{\theta=1} = 1 - E_f\left(r^{-1}\right) \geq 0, \tag{14}$$

which can be written in terms of the density parameters as

$$1 \geq e^{-m+\frac{1}{2}v^2}\left[1+\sum_{s=1}^{n}(-1)^s \delta_s v^s\right] \tag{15}$$

[17] In particular, $(m_{4,WGB2}, ku_{WGB2}) = (0.012407, 4.0828)$, $(m_{4,LN}, ku_{LN}) = (0.01166, 3.83826)$ and $EU_{WGB2} = 0.024281$ and $EU_{LN} = 0.0243951$.

We illustrate how higher-order moments matter when using the log-SNP in comparison with the log-normal case. For $(m_{1,LN}, m_{2,LN}) = (1.05, 0.055125)$ the log-SNP $(n = 3)$ meets condition (15) when it converges to the log-normal with those moments, that is, when $(\delta_1, \delta_2, \delta_3, m_{3,\log-SNP})$ tends to $(0, 0, 0, 0.00881)$. In general, if $(m_{1,\log-SNP}, m_{2,\log-SNP}) = (1.05, 0.055125)$, for values of $m_{3,\log-SNP}$ different from 0.00881 (i.e. the third centered moment of the log-normal for the latter vector of first two centered moments) the log-SNP $(n = 3)$ departs from the log-normal and leads to either *plunging* or *diversifying* when either $m_{3,\log-SNP} > 0.00881$ or $m_{3,\log-SNP} < 0.00881$, respectively.[18] This difference in the investor's choice is, as it was the case in the previous section with the GG case, due to prudence.

For the log-SNP $(n = 4)$ if $(m_{1,LN}, m_{2,LN}, m_{3,LN}) = (1.05, 0.055125, 0.00881)$ and for values of m_4 different from 0.011705 (i.e. the m_4 of the log-normal for the latter vector of first three centered moments), this log-SNP departs from the log-normal and leads to *plunging/non-plunging* when m_4 is smaller/larger than 0.011705.[19] The agent therefore would choose to change her invested share in the risky asset under the former pdf relative to the latter, conditioning on both pdfs having the same first three moments; this agent's behavior is due to the temperance property of her preferences for risk. These results add evidence to the GB2 case on that higher-order moments are relevant for the comparative statics of liquidity preferences.

3 Empirical Application

We illustrate our analysis by assuming an agent faces the choice of allocating wealth between a riskless asset (cash) and a risky asset (S&P500 index). We use data from Robert Shiller's webpage spanning the period January 1871 to February 2011 for a total of one thousand six hundred and eighty two observations. Table 3 presents the descriptive statistics of the gross return series at the monthly frequency computed as $r_t = 1 + \log(P_t/P_{t-1})$, where P_t denotes the real price of the S&P500.

[Insert Table 3 here]

Table 4 provides maximum likelihood estimates of parametric distributions discussed

[18] $EU_{\log-SNP(n=3)}$ with the same $(m_1, m_2) = (1.05, 0.055125)$ $m_3 < 0.00881$ (thus leading to non-plunging) are lower than the EU_{LN} with those moments, i.e. $(m_1, m_2, m_3) = (1.05, 0.055125, 0.00881)$.

[19] $EU_{\log-SNP(n=4)}$ with the same $(m_1, m_2, m_4) = (1.05, 0.055125, 0.00881)$ and $m_4 > 0.011705$ (thus leading to non-plunging) are lower than the EU_{LN} with those moments, i.e. $(m_1, m_2, m_3, m_4) = (1.05, 0.055125, 0.00881, 0.011705)$.

above, three of which are two-parameter distributions (log-normal, gamma, and Weibull); one three-parameter distribution (GG), and one four-parameter distribution (GB2), as well two log-SNP densities, one truncated at two, and the other one truncated at four. All distributions match rather well the first two moments of the return series (with the exception of the Weibull) but there are clear differences in the densities's fit of returns' skewness and kurtosis.[20] The distributions in the application that are most flexible (GB2 and the log-SNP truncated at four) display closer higher order moments to those of the data and present the best fit in terms of log-likelihood and AIC.

The last row in Table 4 indicates if risky-asset-only portfolio conditions are met for each pdf under log-utility. It turns out that for none of the distributions considered the agent would invest all her wealth in the risky asset. This result is in line with the empirical regularity that *plungers* are rarely observed.

[Insert Table 4 here]

4 Conclusions

We examine the issue of the classical portfolio choice theory related with the importance of higher-order moments in the pdf of wealth for the investor decision to diversify or not. We derive the theoretical conditions by which the allocation of all wealth in the risky asset would be optimal for two-, three-, four- and five-parameter densities. Our results show that optimal plunging behavior depends crucially on the higher-order moments of the pdfs, which are associated with higher-order preference properties such as downside risk aversion (or prudence) and temperance.

As an application, we estimate the alternative pdfs on the monthly S&P500 index data from 1871 to 2011. We find that the most general and so flexible pdfs fit better the data and, for none of them the corner solution condition is met, which provides support to the stylized fact that investors do diversify.

[20] The WGB2 estimation yields a non-significant estimate of parameter k, thus converging to the GB2, the latter presenting a better fit according to the AIC as it has less parameters; these results are not presented in Table 2 for the sake of simplicity but are available from the authors upon request.

Appendix A. Plunging with power utility under alternative distributions

We extend the analysis to the power utility function, $u_2(\omega, \lambda) = \omega^\lambda$, $0 < \lambda < 1$, whose risk aversion parameter, λ, is allowed to vary, and it is therefore more general than log utility, u_1.[21] For any pdf, f, normalizing initial wealth ($\overline{\omega}_0$) to 1, the EU is given by,

$$E_f\left[u_2(\omega, \lambda)\right] = E_f(\omega^\lambda) = \int (1 + \theta r - \theta)^\lambda f(r; \mathbf{\Omega}) dr. \tag{16}$$

Differentiating with respect to θ we obtain,

$$\begin{aligned} \xi_f\left(u_2(\omega, \lambda); \theta\right) &= \lambda \int (1 + \theta r - \theta)^{\lambda - 1} (r - 1) f(r; \mathbf{\Omega}) dr \\ &= \lambda \left\{ E_f\left[r\left[(1-\theta) + \theta r\right]^{\lambda-1} \right] - E_f\left[\left[(1-\theta) + \theta r\right]^{\lambda-1} \right] \right\} \end{aligned} \tag{17}$$

Therefore the following two conditions must hold to have a corner solution:

$$\xi_f\left(u_2(\omega, \lambda); \theta\right) > 0 \forall \theta \in [0, 1), \tag{18}$$

$$\left.\xi_f\left(u_2(\omega, \lambda); \theta\right)\right|_{\theta=1} \geq 0. \tag{19}$$

Given the complexity of the solution for a global maximum in this case,[22] we proceed by providing an example where higher-order moments matter for necessary (but not sufficient) condition (19),

$$\begin{aligned} \left.\xi_f\left(u_2(\omega, \lambda); \theta\right)\right|_{\theta=1} &= \lambda\left[E_f\left(r^\lambda\right) - E_f\left(r^{\lambda-1}\right)\right] \geq 0 \\ &= E_f\left[r^\lambda\right] \geq E_f\left[r^{\lambda-1}\right]. \end{aligned} \tag{20}$$

[21] As the exponent of a particular version of the power utility function goes to zero, it becomes the log utility function,

$$\lim_{\lambda \to 0} \frac{\omega^\lambda - 1}{\lambda} = \log(\omega)$$

[22] Note that equation (17) can be rewritten by applying Newton's generalized binomial theorem to obtain the following equation

$$\begin{cases} \lambda \overline{\omega}_0^\lambda \sum_{k=0}^\infty \frac{(\lambda-1)_k}{k!} (1-\theta)^{\lambda-k-1} \theta^k \left\{E_f\left(r^{k+1}\right) - E_f\left(r^k\right)\right\} & \text{if } \frac{1-\theta}{\theta} > r \\ \lambda \overline{\omega}_0^\lambda \sum_{k=0}^\infty \frac{(\lambda-1)_k}{k!} (1-\theta)^k \theta^{\lambda-k-1} \left\{E_f\left(r^{\lambda-k}\right) - E_f\left(r^{\lambda-k-1}\right)\right\} & \text{if } \frac{1-\theta}{\theta} < r \end{cases}$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k-1)$ stands for the Pochhammer's falling factorial. Therefore condition (18) can be expressed as follows: $\xi_f\left(u_2(\omega, \lambda); \theta\right) > 0\ \forall \theta \in [0, 1)$, i.e.,

$$\begin{cases} \sum_{k=0}^\infty \frac{(\lambda-1)_k}{k!} (1-\theta)^{\lambda-k-1} \theta^k \left\{E_f\left(r^{k+1}\right) - E_f\left(r^k\right)\right\} > 0 & \text{if } \frac{1-\theta}{\theta} > r \\ \sum_{k=0}^\infty \frac{(\lambda-1)_k}{k!} (1-\theta)^k \theta^{\lambda-k-1} \left\{E_f\left(r^{\lambda-k}\right) - E_f\left(r^{\lambda-k-1}\right)\right\} > 0 & \text{if } \frac{1-\theta}{\theta} < r \end{cases}$$

We first consider the case of the GB2 for which condition (20) can be written as

$$b^{\lambda}\frac{\Gamma\left(p+\frac{\lambda}{c}\right)\Gamma\left(q-\frac{\lambda}{c}\right)}{\Gamma\left(p\right)\Gamma\left(q\right)}-b^{\lambda-1}\frac{\Gamma\left(p+\frac{\lambda-1}{c}\right)\Gamma\left(q-\frac{(\lambda-1)}{c}\right)}{\Gamma\left(p\right)\Gamma\left(q\right)}\geq 0,$$

and using the expression for b obtained from $\mu_{1,GB2}=\frac{b\Gamma\left(p+\frac{1}{c}\right)\Gamma\left(q-\frac{1}{c}\right)}{\Gamma(p)\Gamma(q)}$, we write this condition as follows

$$\mu_{1,GB2}\geq\frac{\Gamma\left(\frac{pc+1}{c}\right)\Gamma\left(\frac{qc-1}{c}\right)\Gamma\left(\frac{pc+\lambda-1}{c}\right)\Gamma\left(-\frac{-qc+\lambda-1}{c}\right)}{\Gamma\left(\frac{pc+\lambda}{c}\right)\Gamma\left(-\frac{-qc+\lambda}{c}\right)\Gamma\left(p\right)\Gamma\left(q\right)}. \tag{21}$$

For the case of the GG, using its mgf in Table 1, condition (20) can be written as,

$$1-\frac{a\Gamma(p+\frac{\lambda-1}{c})}{\Gamma(p+\frac{\lambda}{c})}\geq 0, \tag{22}$$

and given that $a=\frac{1}{\mu_{1,GG}}\frac{\Gamma(p+\frac{1}{c})}{\Gamma(p)}$, the equation above becomes,

$$\mu_{1,GG}\geq\frac{\Gamma(p+\frac{1}{c})\Gamma(p+\frac{\lambda-1}{c})}{\Gamma(p)\Gamma(p+\frac{\lambda}{c})}. \tag{23}$$

For the case of the two-parameter gamma distribution, condition (23) reduces to[23]

$$1-\frac{a\Gamma(p+\frac{\lambda-1}{c})}{\Gamma(p+\frac{\lambda}{c})}=1-a\frac{1}{p+\lambda-1}\geq 0, \tag{24}$$

which can be expressed in terms of the raw moments as

$$\mu_{1,g}\geq 1+\frac{\mu_{2.g}}{\mu_{1,g}}(1-\lambda). \tag{25}$$

The conditions for a risky-asset-only portfolio above suggest that as the agent becomes more risk averse (lower λ), she is less likely to allocate all her wealth to risky assets. Furthermore, for the log utility the investor is less likely to plunge ($\lambda=0$) and it sets an upper bound for the plunging condition under power utility.

For the log-SNP case we obtain condition (19) from the mgf (equation (36)) as

$$e^{\lambda m+\frac{1}{2}\lambda^2 v^2}\left[1+\sum_{s=1}^{n}\delta_s(v\lambda)^s\right]\geq e^{(\lambda-1)m+\frac{1}{2}(\lambda-1)^2v^2}\left[1+\sum_{s=1}^{n}\delta_s(v(\lambda-1))^s\right]. \tag{26}$$

We note that when $\delta_s=0$ for all s in the equation above, we obtain condition (19) for the log-normal distribution.

[23] This expression is also obtained in Boyle and Conniffe (2005).

Table A.1 illustrates our results by giving an example about how the condition for non-diversifyers does depend on higher-order moments, assuming a coefficient of risk aversion of $\lambda = 0.8$. These results suggest that, if returns are characterized by a gamma distribution, condition (25) would not be met, and therefore, it would not be optimal for the agent to *plunge*. However, under the GB2 and log-SNP (n=3) distributions with the same first and second central moments but higher third moment than the gamma, we find that allocating all wealth to the risky asset would be optimal as the agent's risk preferences exhibit prudence.[24] Furthermore, we demonstrate that the fourth moment switches the agent's decision away from the corner solution by considering a log-SNP ($n = 4$) which differs from the log-SNP ($n = 3$) only in m_4, because of temperance in the investor's preferences for risk.

[Insert Table A.1]

Appendix B. Plunging condition for the log-normal

The log-normal pdf assumes that the logarithm of the risky asset (gross) return, $\ln(r)$, follows a Normal distribution with parameters m and v as

$$\Phi(r; m, v) = \frac{1}{rv\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(r)-m}{v}\right)^2},\ 0 < r < \infty. \tag{27}$$

The raw moments (mgf) of this distribution are given by

$$\mu_{t,LN} = E_\Phi[r^t] = \int r^t \Phi(r; m, v) dr = e^{tm+\frac{1}{2}t^2v^2},\ \forall t \in \mathbb{R} \text{ or } \forall t \in \mathbb{C}. \tag{28}$$

$\theta = 1$ is optimum if the condition below holds,

$$\xi_\Phi(u_1(\omega); \theta; m, \nu)|_{\theta=1} = 1 - E_\Phi\left(r^{-1}\right) \geq 0. \tag{29}$$

which is expressed as

$$\begin{aligned} 1 &\geq e^{-m+\frac{1}{2}v^2}, \\ m &\geq (\frac{1}{2})v^2. \end{aligned} \tag{30}$$

Given that $\mu_{1,LN} = e^{m+\frac{1}{2}v^2}$ and $\mu_{2,LN} = e^{2m+2v^2}$ the condition above is: $2\ln\mu_{1,LN} - \frac{1}{2}\ln\mu_{2,LN} \geq \frac{1}{2}\ln\mu_{2,LN} - \ln\mu_{1,LN}$, or $3\ln\mu_{1,LN} \geq \ln\mu_{2,LN}$, so we can write the condition

[24] Within the four-parameter distribution GB2, it is also possible to show that a different parameterization such as $(p, c, q, b) = (1.17620698963, 2, 6.1, 2.485)$ yields the same mean and variance but lower skewness ($m_3 = 0.19171$) and condition (21) would not be met.

for the corner solution in terms of either the parameters (equation (30)), the first two raw moments (equation (31)) or the central moments (equation (32)),[25]

$$\mu_{1,LN} \geq \left(1 + \frac{\mu_{2,LN} - \mu_{1,LN}^2}{\mu_{1,LN}^2}\right) \quad (31)$$

$$m_{1,LN} \geq 1 + \frac{m_{2,LN}}{m_{1,LN}^2} \quad (32)$$

Appendix C. Log-SNP

If r follows a log-SNP truncated at order n, then the following pdf holds,[26]

$$\begin{aligned} \Upsilon_n\left(r; m, v, \boldsymbol{\delta}\right) &= \left[1 + \sum_{s=1}^{n} \delta_s C_s(x)\right] \Phi(r, m, v), \\ \Phi(r; m, v) &= \frac{1}{vr\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{rv}\phi(x), \\ x &= \frac{\ln(r) - m}{v}, \ 0 < r < \infty. \end{aligned} \quad (33)$$

where $\boldsymbol{\delta} = (\delta_1, ..., \delta_n)'$ is the vector of density parameters, $\phi(\cdot)$ stands for the standard Normal pdf and $C_s(x)$ is the sth order Chebyshev-Hermite polynomial, which can be defined by the identity in equation (34),

$$\frac{d^s\phi(x)}{dx^s} = (-1)^s C_s(x)\phi(x), \ \forall s \geq 1. \quad (34)$$

This distribution inherits all the good properties of the SNP approach based on Gram-Charlier series, namely:

1. Generality: not only it encompasses the log-normal but it can also approximate any pdf to any desirable degree of accuracy depending on the truncation order n;

2. Flexibility: it is endowed with a variable number of parameters to capture whatever moment structure;

3. Orthogonality: Chebyshev-Hermite polynomials form an orthonormal basis with respect to the weight function $\phi(x)$, equation (35), which makes the specification very

[25] It is worth noting that in Boyle and Coniffe (2005) the same expression is obtained through the Taylor approximation for r^{-1}.

[26] Note that the log-SNP distribution is a log-linear transformation of a truncated Gram-Charlier Type A expansion.

tractable.[27]

$$\int_{-\infty}^{\infty} C_s(x)C_r(x)\phi(x)dx = \begin{cases} 0, & s \neq r \\ s!, & s = r. \end{cases} \tag{35}$$

The statistical properties of the log-SNP can be straightforwardly derived from those of the log-normal. For example, it is easily checked that equation (33) defines a density function (i.e. it integrates up to one; see Proof 1). Also, its raw moments can be obtained from the mgf of the Gram-Charlier distribution, $M_x(t)$, as displayed in equation (36) (see Proof 2).

$$\mu_{t,LSNP} = E_{\Upsilon_n}[r^t] = e^{mt}M_x(vt) = e^{mt+\frac{1}{2}t^2v^2}\left[1+\sum_{s=1}^{n}\delta_s(vt)^s\right], \tag{36}$$

It is noteworthy that the moments of the log-SNP are computed directly from the Gram-Charlier's mgf, unlike the moments of the Gram-Charlier density that are obtained from the derivatives of its mgf. Therefore the moments of the log-SNP depend on the whole parametric structure of the density. Conversely, the parameter δ_s is obtained as a linear combination of the first s raw moments of the log-SNP distribution as in equation (37),

$$\delta_i = c_{0i} + \sum_{t=1}^{n} c_{ti}\mu_t, \tag{37}$$

where $\{c_{ti}\}_{t=0}^{n}$ is the sequence of constants of every raw moment in parameter δ_i.

Proof 1. The log-SNP density integrates up to one.

$$\begin{aligned} \int \Upsilon_n\left(r; m, v, \boldsymbol{\delta}\right) dr &= \int_0^{\infty}\left[1+\sum_{s=1}^{n}\delta_s H_s\left(\frac{\ln(r)-m}{v}\right)\right]\phi\left(\frac{\ln(r)-m}{v}\right)\frac{1}{rv}dr \\ &= \int_{-\infty}^{\infty}\left[1+\sum_{s=1}^{n}\delta_s H_s(x)\right]\phi(x)dx = 1. \end{aligned} \tag{38}$$

Proof 2. The moments of the log-SNP distribution can be obtained through the mgf of the Gram-Charlier distribution,$M_x(t)$.

$$\begin{aligned} \mu_{t,LSNP} &= E_{\Upsilon_n}[r^t] = \int_0^{\infty} r^t\left[1+\sum_{s=1}^{n}\delta_s H_s\left(\frac{\ln(r)-m}{v}\right)\right]\Phi(r,m,v))dr \\ &= \int_{-\infty}^{\infty} e^{(vx+m)t}\left[1+\sum_{s=1}^{n}\delta_s H_s(x)\right]\phi(x)dx \\ &= E_f\left[e^{(vx+m)t}\right] = E_f\left[e^{mt}e^{vxt}\right] = e^{mt}E_f\left[e^{vtx}\right] \\ &= e^{mt}M_x(vt), \end{aligned} \tag{39}$$

[27] See Abramowitz and Stegun (1972) or Kendall and Stuart (1977) for further details on Gram-Charlier Series properties.

where $M_x(vt)$ is,

$$\begin{aligned}
M_x(t) &= \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} e^{tx}\phi(x)dx + \sum_{s=1}^{n}\delta_s \int_{-\infty}^{\infty} e^{tx} H_s(x)\phi(x)dx \\
&= e^{t^2/2} + \sum_{s=1}^{n}\delta_s \left[-e^{tx} H_{s-1}(x_t)\phi(x_t)\,\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} te^{tx} H_{s-1}(x_t)\phi(x_t)dx \right] \\
&= e^{t^2/2} + \sum_{s=1}^{n}\delta_s \int_{-\infty}^{\infty} t^s e^{tx}\phi(x)dx = e^{t^2/2}\left[1 + \sum_{s=1}^{n}\delta_s t^s \right]. \qquad (40)
\end{aligned}$$

Integrating by parts and taking into account that $\frac{dH_s(x)}{dx} = sH_{s-1}(x)$ and $e^{tx}H_s(x)\phi(x) \underset{x\to\pm\infty}{\longrightarrow} 0$, $\forall s \geq 1$.

$$\begin{aligned}
u &= e^{tx} \Longrightarrow du = te^{tx}dx \\
dv &= H_s(x)\phi(x)dx \Longrightarrow v = -H_{s-1}(x)\phi(x). \qquad (41)
\end{aligned}$$

References

[1] Adler, M. 1969. On the Risk-Return Trade-Off in the Valuation of Assets. *Journal of Financial and Quantitative Analysis* 4, 493-512.

[2] Abramowitz, M., and Stegun, I. A. 1972. Orthogonal Polynomials. In M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*. New York: Dover.

[3] Arditti, F. D. 1967. Risk and the Required Return on Equity. *Journal of Finance* 22, 19-36.

[4] Arditti, F. D., and Levy, H. 1975. Portfolio Efficiency Analysis in Three Moments: The Multiperiod Case. *Journal of Finance* 30, 797-809.

[5] Bookstaber, R. M., and McDonald, J. B. 1987. A General Distribution for Describing Security Price Returns. *The Journal of Business* 60, 401-424.

[6] Bombardini, M., and Trebbi, F. 2012. Risk Aversion and Expected Utility Theory: A Field Experiment with Large and Small Stakes. *Journal of the European Economic Association* 10, 1348-1399.

[7] Borch, K. 1969. A Note on Uncertainty and Indifference Curves. *The Review of Economic Studies* 36, 1-4.

[8] Boyle, G. and Conniffe, D. 2005. Compatibility of Expected Utility and μ/σ Approached to Risk for a class of Non Location-Scale Distributions. *Economic Theory* 35, 336-343.

[9] Boyle, G. and Conniffe, D. 2005. When does 'All Eggs in One Risky Basket' Make Sense? Proceedings of the 2005 *Irish Economics Association's Annual Conference.*

[10] Brandt, M., Goyal A., Santa-Clara, P., and Stroud, J. 2005. A Simulation Approach to Dynamic Portfolio Choice with an Application to Learning about Return Predictability. *Review of Financial Studies* 18, 831–873.

[11] Chambers, R. G., and Quiggin, J. 2007. Dual Approaches to the Analysis of Risk Aversion. *Economica* 74, 189-213.

[12] Charlier, C. V. L. 1906. Uber Die Darstellung Willkurlicher Funktionen. *Arvik fur Mathematik Astronomi och Fysik*, 2, 20, 1-35.

[13] Chebyshev, P. L. 1890. Sur Deux Théorèmes Relatifs aux Probabilités. *Acta Mathematica* 14, 305-315.

[14] Chetty, R. 2006. A New Method of Estimating Risk Aversion. *American Economic Review* 96, 1821-1834.

[15] Chunhachinda, P., Dandapani, K., Hamid, S., and Prakash, A. J. 1997. Portfolio Selection and Skewness: Evidence from International Stock Markets. *Journal of Banking and Finance* 21, 143–167.

[16] Corrado, C., and Su, T. 1996. Skewness and Kurtosis in S&P 500 Index Returns Implied by Option Prices. *Journal of Financial Research* 19, 175-192.

[17] Del Brio, E., Ñíguez, T. M., and Perote, J. 2011. Multivariate Semi- Nonparametric Distributions with Dynamic Conditional Correlations. *International Journal of Forecasting* 27, 347-364.

[18] Edgeworth, F. Y. 1896. The Asymmetrical Probability Curve. *Philosophical Magazine,* Series 5, 41, 249, 90-99.

[19] Edgeworth, F. Y. 1907. On the Representation of Statistical Frequency by a Series. *Journal of the Royal Statistical Society* 70, 102-106.

[20] Eeckhoudt, L. and Schlesinger, H. 2006. Putting Risk in its Proper Place. *American Economic Review* 96, 280-289.

[21] Eichner, T., and Wagener, A. 2011. Increases in Skewness and Three-Moment Preferences. *Mathematical Social Sciences* 61, 109-113.

[22] Fama, E. F. 1965. The Behavior of Stock Market Prices. *Journal of Business* 38, 34-105.

[23] Feldstein, M. S. 1969. Mean-Variance Analysis in the Theory of the Firm under Uncertainty. *The Review of Economic Studies* 36, 5-12.

[24] Feldstein, M. S. 1978. A Note on Feldstein's Criticism of Mean-Variance Analysis: A Reply. *The Review of Economic Studies* 45, 201.

[25] Gallant, A. R., and Nychka, D. W. 1987. Seminonparametric Maximum Likelihood Estimation. *Econometrica* 55, 363-390.

[26] Goldman, M. B. 1979. Anti-Diversification or Optimal Programmes for Infrequently Revised Portfolios. *Journal of Finance* 34, 505-516.

[27] Gollier, C. 1995. The Comparative Statics of Changes in Risk Revisited. *Journal of Economic Theory* 66, 522-535.

[28] Gram, J. P. 1883. Über die Entwicklung reeler Funktionen in Reihen mittelst der Methode der kleinsten Quadrate. *Journal für die reine und angewandte Mathematik* 94, 41-73.

[29] Glustoff, E. and Nigro, N. 1972. Liquidity Preference and Risk Aversion with an Exponential Utility Function. *The Review of Economic Studies* 39, 113-115.

[30] Haliassos, M., and Betaut, C. C. 1995. Why Do so Few Hold Stocks? *The Economic Journal* 105, 1110-1129.

[31] Horvath, P. A., and Scott, R. C. 1985. An Expected Utility Explanation of Plunging and Dumping Behavior. *Financial Review* 20, 219-228.

[32] Jarrow, R., and Rudd, A. 1982. Approximate Option Valuation for Arbitrary Stochastic Processes. *Journal of Financial Economics* 10, 347-369.

[33] Jensen, H. M., Johansen, A., and Simonsen I. 2003. Inverse Statistics in Economics: The Gain–Loss Asymmetry. Physica A 324, 338-343.

[34] Jondeau, E., and Rockinger, M. 2001. Gram-Charlier Densities. *Journal of Economic Dynamics and Control* 25, 1457-1483.

[35] Jondeau, E., and Rockinger, M. 2006. Optimal Portfolio Allocation Under Higher Moments. *Journal of the European Financial Management Association* 12, 29–67.

[36] Jurczenko, E., and Maillet, B. 2006. Theoretical Foundations of Asset Allocation and Pricing Models with Higher-Order Moments, in *Multi-moment Asset Allocation and Pricing Models*, edited by E. Jurczenko and B. Maillet. John Wiley & Sons, Ltd.

[37] Kane, A. 1982. Preference and Portfolio Choice. *Journal of Financial and Quantitative Analysis* 17, 15-25.

[38] Kendall, M., and Stuart, A. 1977. *The Advanced Theory of Statistics*, London: Charles Griffing & Co Ltd.

[39] Kimball, M. S. 1990. Precautionary saving in the Small and in the Large. *Econometrica* 58, 53–73.

[40] Kimball, M. S. 1992. Precautionary Motives for Holding Assets, in *New Palgrave Dictionary of Money and Finance,* edited by P. Newman, M. Milgate and J. Falwell. London: MacMillan.

[41] Kraus, A., and Litzenberger, R. 1976. Skewness Preference and the Valuation of Risk Assets. *Journal of Finance* 31, 1085-1100.

[42] Lajeri-Chaherli, F. 2004. Proper Prudence, Standard Prudence and Precautionary Vulnerability. *Economics Letters* 82, 29–34.

[43] León, A., Mencía, J., and Sentana, E. 2009. Parametric Properties of Semi-Nonparametric Distributions, with Applications to Option Valuation. *Journal of Business and Economic Statistics* 27, 176-192.

[44] Levy, H. 1969. A Utility Function Depending on the First Three Moments. *Journal of Finance* 24, 715-719.

[45] Mandelbrot, B. 1963. The Variation of Certain Speculative Prices. *The Journal of Business* 36, 1279-1313.

[46] Markowitz, H., 1952. Portfolio Selection. *Journal of Finance* 7, 7-91.

[47] Mayshar, J. 1978. A Note on Feldstein's Criticism on Mean-Variance Analysis. *Review of Economic Studies* 45, 197-199.

[48] McDonald, J. B. 1984. Some Generalized Functions for the Size Distribution of Income. *Econometrica* 52, 647-663.

[49] McDonald, J. B., and Xu, Y.J. 1995. A Generalization of the Beta Distribution with Applications. *Journal of Econometrics* 66, 133-152.

[50] Menezes, C., Geiss, C. and Tressler, J. 1980. Increasing Downside Risk. *American Economic Review* 70, 921–32.

[51] Meyer, J. 1987. Two-Moment Decision Models and Expected Utility Maximization. *American Economic Review* 77, 421-430.

[52] Miller, S. M. 1975. Measures of Risk Aversion: Some Clarifying Comments. *Journal of Financial and Quantitative Analysis* 10 (2), 299-309.

[53] Mittnik, S., and Rachev, S. 1993. Reply to Comments on "Modelling Asset Returns with Alternative Stable Model" and Some Extensions. *Econometric Reviews* 12, 347-389.

[54] Ñíguez, T. M., Paya, I., Peel, D. A., and Perote, J. 2012. On the Stability of the Constant Relative Risk Aversion (CRRA) Utility Under High Degrees of Uncertainty. *Economics Letters* 115, 244-248.

[55] Ñíguez, T. M., and Perote, J. 2012. Forecasting Heavy-Tailed Densities with Positive Edgeworth and Gram-Charlier Expansions. *Oxford Bulletin of Economics and Statistics* 74, 620-627.

[56] Ormiston, M. B., and Schlee, E. 2001. Mean-variance Preferences and Investor Behaviour. *Economic Journal* 111, 849-861.

[57] Patton, A. 2006. Modelling Asymmetric Exchange Rate Dependence. *International Economic Review* 47, 527-556.

[58] Prakash, A. J., Chang, C. H., and Pactwa, T. E. 2003. Selecting a Portfolio with Skewness: Recent Evidence from US, European and Latin American Equity Markets. *Journal of Banking and Finance* 27, 1375-1390.

[59] Sargan, J. D. 1975. Gram-Charlier Approximations Applied to T Ratios of K-Class Estimators. *Econometrica* 43, 327-346.

[60] Simkowitz, M., and Beedles W. 1978. Diversification in a Three Moment World. *Journal of Financial and Quantitative Analysis* 13, 927-941.

[61] Tobin, J. E. 1958. Liquidity Preference as Behavior toward Risk. *The Review of Economic Studies* 25, 65-68.

[62] Tobin, J. E. 1969. Comment on Borch and Feldstein. *The Review of Economic Studies* 36, 13-14.

[63] Yaari, M. 1987. The Dual Theory of Choice Under Risk. *Econometrica* 55, 95–115.

[64] Ye, Y., Oluyede, B. O., and Pararai, M. 2012. Weighted Generalized Beta Distribution of the Second Kind and Related Distributions. *Journal of Statistical and Econometric Methods* 1, 13-31.

Figures

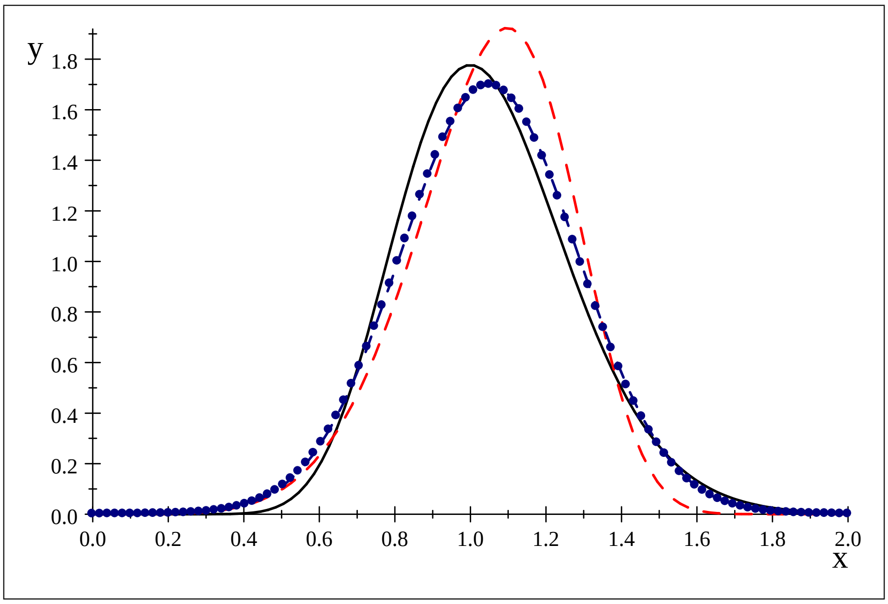

Figure 1. Probability density funciton of the log-normal (black solid line), Weibull (red dash line), and Gamma (blue dot-dash line).

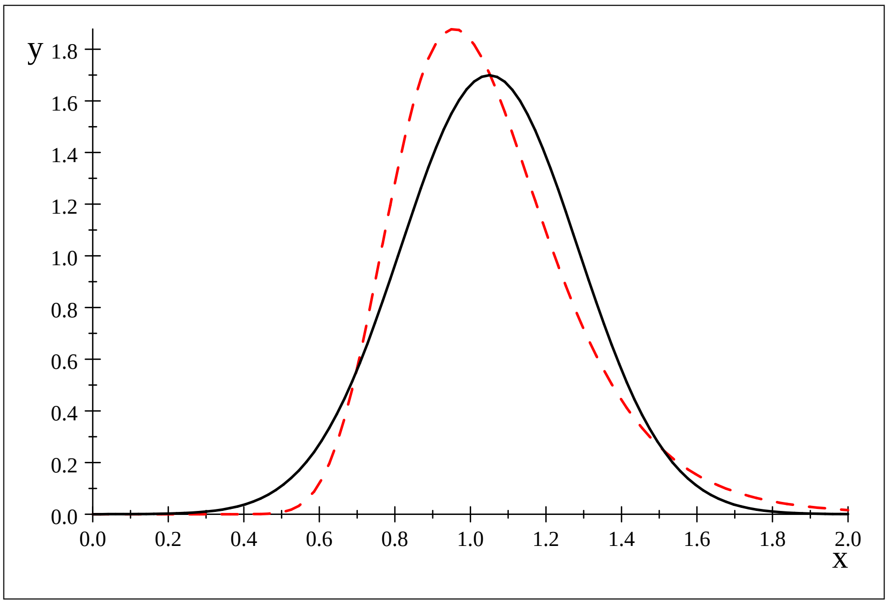

Figure 2. Probability density function of the log-normal (Black solid line), and GB2 (red dash line).

Tables

TABLE 1

Density and moment generating functions of the generalized distributions

	pdf	$mgf = E[r^t]$
$WGB2(r;k,c,b,p,q)$	$\frac{cr^{cp+k-1}\Gamma(p+q)}{b^{cp+k}\Gamma\left(p+\frac{k}{c}\right)\Gamma\left(q-\frac{k}{c}\right)(1+\frac{r^c}{b^c})^{p+q}}$	$\frac{b^t\Gamma(p+\frac{k}{c}+\frac{t}{c})\Gamma(q-\frac{k}{c}-\frac{t}{c})}{\Gamma(p+\frac{k}{c})\Gamma(q-\frac{k}{c})}$
$GB2(r;c,b,p,q)$	$\frac{cr^{cp-1}\Gamma(p+q)}{b^{cp}\Gamma(p)\Gamma(q)(1+\frac{r^c}{b^c})^{p+q}}$	$\frac{b^t\Gamma\left(p+\frac{t}{c}\right)\Gamma\left(q-\frac{t}{c}\right)}{\Gamma(p)\Gamma(q)}$
$GG(r;a,p,c)$	$\frac{ca^{cp}r^{cp-1}e^{-(ar)^c}}{\Gamma(p)}$	$\frac{1}{a^t}\frac{\Gamma(p+\frac{t}{c})}{\Gamma(p)}$

Notes: Pdfs and mgfs of WGB2, GB2 and GG distributions. $\Gamma(p) = \int_0^\infty e^{-r}r^{p-1}dr$ denotes the gamma function. Parameter k controls the shape and skewness of the WGB2 density, which nests the GB2 when $k = 0$ (Ye *et al.*, 2012), which, in turns, nests the GG when $b = a^{-1}q^{\frac{1}{c}}$ as $q \to \infty$ (McDonald, 1984).

TABLE 2

GB2-class of densities and *plunging*: Range of moments and *pdf* specification

	GB2	GG	gamma	Weibull	LN
Panel A. Maximum m_2 for which PC holds within a class of pdf					
m_2^*	0.0581	0.055125	0.05250	0.043580	0.055125
m_3	0.0141	0.008826	0.00525	-0.00323	0.008826
PC	Yes	Yes	Yes	Yes	Yes
Panel B. Examples of GG distributions with same (m_1, m_2) as log-normal in Panel A					
GG that nests LN: $(c, p, a) = (2, 5.11592, 2.1022)$					
m_2		0.055125			0.055125
m_3		0.003034			0.008826
PC		No			Yes
GG that nests LN: $(c, p, a) = (0.81694, 30, 61.503)$					
m_2		0.055125			0.055125
m_3		0.006324			0.008826
PC		No			Yes
Panel C. Examples where general pdf matters for exact values of (m_2, m_3) in PC					
GB2 that nests Weibull: $(p, c, q, b) = (1, 5.855105, 90, 2.4414)$					
m_2	0.04368			0.043580	
m_3	-0.0031			-0.00323	
PC	Yes			Yes	
GB2 that nests gamma: $(p, c, q, b) = (21.55, 1, 791, 38.492)$					
m_2	0.0526		0.05250		
m_3	0.0054		0.00525		
PC	Yes		Yes		
Panel D. Example of two GB2 pdfs with same (m_1, sk) and PC holds for higher m_2					
GB2 with $(m_1, sk) = (1.05, 2.194406)$					
p	1.77451	22			
c	7.574	4.5			
q	0.85	1.5			
b	0.88208	0.52162			
m_2	0.06819	0.06836			
PC	No	Yes			

Notes: Summary of the plunging condition (PC) examples for the GB2-class of distributions presented in Section 2.1. For all cases $m_1 = 1.05$. m_2^* denotes the maximum variance so that PC holds.

TABLE 3

Monthly log gross returns descriptive statistics

	S&P500
Sample	02/1871 − 02/2011
Observations	1681
Mean	1.00167
Median	1.00525
Maximum	1.41480
Minimum	0.69247
St. Dev.	0.041352
Skewness	−0.30782
Kurtosis	13.9528
Jarque-Bera	8429.14*

Notes: The Jarque-Bera normality test is asymptotically distributed as a $\chi^2(2)$ under the null of normality. The critical values of $\chi^2(2)$ is 5.99 at 5% significance level, respectively. The asterisk (*) denotes that the null hypothesis of the test is rejected at least at 5% significance level.

TABLE 4
Estimation results, S&P500 01/1871-02/2011

	GB2	GG	gamma	Weibull	LN	log-SNP (n=2)	log-SNP (n=4)
$\widehat{m}$					0.0008	−0.0368	−0.0643
					(0.78)	(−20.9)	(−19.2)
$\widehat{v}$					0.0420	0.0425	0.0431
					(57.9)	(48.1)	(34.1)
$\widehat{b}$	1.017						
	(9.07)						
$\widehat{q}$	0.550						
	(2.86)						
$\widehat{c}$	98.71	4.086		16.15			
	(6.05)	(0.15)		(84.2)			
$\widehat{p}$	0.314	33.75	575.05				
	(6.03)	(3.03)	(70.3)				
$\widehat{a}$		2.356	574.049	0.979			
		(0.13)	(69.2)	(627.2)			
$\widehat{\delta}_1$						0.8852	1.5098
						(41.9)	(38.4)
$\widehat{\delta}_2$						0.3776	1.1128
						(16.1)	(19.6)
$\widehat{\delta}_3$							0.3900
							(12.8)
$\widehat{\delta}_4$							0.0875
							(7.17)
$\widehat{m}_1$	1.0014	1.0015	1.0016	0.9876	1.0017	1.0017	1.0017
$\widehat{m}_2$	0.00154	0.00179	0.00174	0.00567	0.00177	0.00172	0.00170
$\widehat{sk}$	−0.6349	−0.0452	0.0834	−0.8097	0.1260	−0.4383	−0.6752
$\widehat{m}_3$	−0.000038	−0.000003	0.000006	−0.000306	0.000009	−0.000031	−0.000047
$\widehat{ku}$	5.2036	3.0035	3.0104	4.0684	3.0282	4.3687	5.2827
$\widehat{m}_4$	0.000012	0.0000096	0.0000091	0.000111	0.0000095	0.000013	0.0000153
LogL	3142.4	2967.9	2953.8	2464.8	2942.9	3050.4	3075.2
AIC	−3.7340	−3.5275	−3.5120	−2.9278	−3.4991	−3.6245	−3.6528
PC	No	No	No	No	No	No	No

Notes: Estimation results (ML t-statistics in brackets) for the GB2, GG, gamma, Weibull, log-normal (LN) and log-SNP (n) distributions. n denotes the log-SNP truncation order. m_i denote central moment of order i, sk and ku denote skewness and kurtosis, respectively. AIC and logL denote Akaike Information Criterion and log likelihood, respectively. PC denotes whether plunging condition is met.

TABLE A.1

Plunging condition (19) under different distributions

	GB2	gamma	log-SNP (n=3)	log-SNP (n=4)
m			-0.079225	-0.079225
v			0.50599	0.50599
b	0.5764			
q	6.1			
c	1.1			
p	10	3.426730915		
a		3.263553253		
δ_1			0.024413	-0.10808
δ_2			-0.072362	0.40769
δ_3			0.047676	-0.46982
δ_4				0.17046
m_1	1.05	1.05	1.05	1.05
m_2	0.32174	0.32174	0.32174	0.32174
m_3	0.41694	0.19717	0.41692	0.41692
m_4	1.62705	0.49178	1.39080	2.90540
PC	Yes	No	Yes	No

Notes: This table presents whether the condition (19) (denoted as PC) is met for the GB2, gamma and log-SNP distributions for different values of parameters so that the pdfs yield the same first two central moments and differ on the third and/or the fourth moment.