AN INTRODUCTION TO ASYMPTOTIC SERIES

A PROJECT REPORT SUBMITTED

BY

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Abstract

Asymptotic expansions, or asymptotic series, play a major role in modern mathematics and physics. However, the subject itself is quite old, dating back from the end of the 19^{th} century. The general idea of these methods is to represent complicated functions as sums or products of simpler ones and truncating the series after a finite number of terms being used, usually when the independent variable tends to a specific value or infinity. If the finite series seems to give good approximations of the initial functions, then the question of whether the approximations are satisfactory arises. Increasing the number of operations generally, but not always, produces finer results, but having too many terms can make the calculation hard. That is why we need a method of classifying how accurate our approximations are. Saying that asymptotic series deal with problems that are only part of the discussion above is wrong. The question of what they really are is hard to answer. The reason why we do not often see such terms in other branches of mathematics is probably because they have already been named differently, for example Integral Calculus.

The goal of this project is to explore the idea behind the asymptotic series by classifying them formally. It starts with a review of differential equations and how asymptotic methods are necessary to be introduced and continues with chapter two where several important properties are proved. At the end, a simple example is explored in detail, namely the Watson's lemma, which is vital for many other branches of the asymptotic methods, such as the Laplace's method for integrals and the method of steepest descent.

Contents

1 Introduction

Definition 1.1. *A mathematical expression is said to be in closed form if it can be written as a finite number of operations. It may contain constants, variables, "well-known" operations (e.g.,* $\pm$*,* $\times$*,* $\div$*), and functions (e.g.,* n^{th} *root, exponent, logarithm, trigonometric functions etc.), but usually no limits or integrals.*

In the theory of differential equations we are very often unable to solve some equations in closed form. Therefore expressing their solutions in terms of known functions might not possible. We provide two simple examples:

Example 1.1. *Consider the infinite sum:*

$$\lim_{k\to\infty}\sum_{n=0}^{k}\frac{x}{5^n}=\sum_{n=0}^{\infty}\frac{x}{5^n}=x\sum_{n=0}^{\infty}(\frac{1}{5})^n=\frac{5}{4}x. \tag{1.1}$$

x is there simply to illustrate that the expression can depend to a certain variable. We assume that x is finite and use the fact that the geometric series is convergent, since $1/5 < 1$. Hence, the infinite series converges to the limit of partial sums: $S_k(x)=\sum_{n=0}^{k}\frac{x}{5^n}$, as k goes to infinity. Therefore, we can apply properties of convergent sums and take x out of the summation (being the constant sequence for a fixed x), since it is independent of the summed index. Now, we say that the LHS of (1.1) is not in closed form because the summation entails an **infinite** number of elementary operations. However, the RHS of (1.1) is a linear function of x and is in closed form by Def. 1.1.

Example 1.2. *The error function, which is (up to constants) the antiderivative of the elementary function* e^{-x^2}*, is not in closed form:*

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du. \tag{1.2}$$

The reason for this is because the integral cannot be expressed in terms of finite applications of known functions. It is important to understand that it is not the integral itself that prevents us from expressing the error function in closed form but the fact that it there does not exist a finite composition of known functions that gives the value of the integral for a fixed x (as we know from an elementary course on PDEs).

1.1 Background material

Let us first introduce some useful concepts.

Definition 1.2. *Let G be an open set in* $\mathbb{C}$ *and* $f : G \to \mathbb{C}$*. Then f is analytic (holomorphic) if it is continuously differentiable on G. [1]*

Lemma 1.1. *(Variation of the Estimation lemma) Let C be a contour in $\mathbb{C}$ and let $f : C \to \mathbb{C}$ be a continuous function on C. Suppose also that C lies within a compact (closed and bounded) subset of $\mathbb{C}$. Then $\|f\|_\infty := \sup_{z\in C} |f(z)|$ is achieved for some "$z \in C$"*[1] *and the following holds:*

$$\Big| \int_C f(z)dz \Big| \le \|f\|_\infty \, L(C), \tag{1.3}$$

where $L(C)$ is the length of the contour defined in the usual way by a parametrization function $g : [a, b] \to \mathbb{C}$ from some interval $[a, b]$ in $\mathbb{R}$ to $\mathbb{C}$ (in our case $g(a) = g(b)$ since the contour is closed), such that $x(t) = Re(g(t))$ and $y(t) = Im(g(t))$ have continuous derivatives on each subinterval (t_k, t_{k+1}) for $k = 0, .., n-1$, where $a = t_0 < t_1 <, ..., < t_n = b$ and:

$$L(C) := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \sqrt{(x'(t))^2 + (y'(t))^2} dx \in \mathbb{R}_{\ge 0}. \tag{1.4}$$

The proof of Lem. 1.1 can be found in almost all complex analysis textbooks. It is worth mentioning that there is a difference in how we define the length of a contour in complex and real analysis. We clearly see that the parametrization function g should be continuously differentiable on every subinterval. In real analysis, however, we would require it to be injective as well. The reason is because formula (1.4) actually computes the "path" travelled along the curve. In complex analysis it does not matter if we move forward, backward and forward again along the contour as integrating in an opposite direction simply produces the same result but with a different sign, hence the two contributions cancel, so the "path" travelled and the length coincide. In $\mathbb{R}^n$ this is clearly not true, therefore the additional requirement is necessary. We also note that our notation above is intuitive rather than mathematically correct. For instance, $\|f\|_\infty$ comes from the supremum metric, but we have not defined our metric space. To see why this could be a problem, let us restrict to real-valued functions and consider the vector space of all continuous functions over the closed interval $[a, b] \subset \mathbb{R}$ (i.e. $\mathscr{C}[a, b]$). Then, from an elementary course on metric spaces, we know that the metric space $(\mathscr{C}[a, b], d_\infty)$ is complete (i.e. every Cauchy sequence of functions from $\mathscr{C}[a, b]$ converges (uniformly) to a function in $\mathscr{C}[a, b]$), where the supremum metric d_∞ is defined by:

$$d_\infty(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|, \quad \forall f, g \in \mathscr{C}[a, b]. \tag{1.5}$$

However, the same vector space equipped with the d_1 metric defined by:

$$d_1 := \int_a^b |f(x) - g(x)|dx, \quad \forall f, g \in \mathscr{C}[a, b] \tag{1.6}$$

is not complete. We say that those two metric spaces (as we have different metrics even though the set is the same) are not topologically equivalent. To understand why this is the case, please consult

[1]The quotation marks refer to something said is a sloppy way. In this case, saying that $z \in C$ is not mathematically correct, since the contour also entails its parametrization and using a different one will change the contour, even if the image points stay the same. We should say that z belongs to the set formed by the image of the parametrization function, but this is a minor detail, though worth pointing out.

[8] on page 70 (example 6.36). This means that we have to be careful when working with metric spaces. For more information about how these concepts are applied to functions defined over the complex plane, consult [7] (chapter 3 - page 61).

A slight note on the d_1 metric. As used above, the metric comes from the $L^1 - norm$. However, at a later stage, we are going to use the d_1 metric over $\mathbb{R}^n$, which takes the simpler form:

$$d_1(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} |x_i - y_i|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{1.7}$$

In particular, when the space is only $\mathbb{R}$, then the d_1 metric is simply the absolute value one (also called the standard metric).

Theorem 1.1. *Let C in $\mathbb{C}$ be a contour and let $f_1, f_2, ...$ be a sequence of functions defined and continuous on C. We assume that $f_1, f_2, ...$ uniformly converge to a (continuous) function f. Then:*

$$\lim_{m\to\infty} \int_C f_m(z)dz = \int_C f(z)dz. \tag{1.8}$$

Proof. We propose the following proof: By definition of uniform convergence, we have that $\forall\delta > 0$ $(\delta \in \mathbb{R})$ $\exists m_0 \in \mathbb{N}$, such that $\forall m > m_0$ $(m \in \mathbb{N})$ we have $|f_m(z) - f(z)| < \frac{\delta}{2}$ for all "$z \in C$". Then, it must hold that $\sup_{z\in C} |f_m(z) - f(z)| \leq \frac{\delta}{2} < \delta$. Now, let $\varepsilon > 0$ be arbitrary and choose $\delta = \frac{\varepsilon}{L(c)}$.

Then, consider the following:

$$\Big| \int_C f_m(z)dz - \int_C f(z)dz \Big| = \Big| \int_C (f_m(z) - f(z))dz \Big| \tag{1.9}$$

$$\leq \|f - g\|_\infty L(C) = L(C) \sup_{z\in C} |f_m(z) - f(z)| \quad \text{(by Lem. 1.1)} \tag{1.10}$$

$$< L(C)\delta = L(C)\frac{\varepsilon}{L(C)} = \varepsilon \tag{1.11}$$

Since ε was arbitrary, the inequality must hold for all $\varepsilon > 0$, which proves the theorem. □

Theorem 1.2. *Let a_0, a_1,...$\in \mathbb{C}$ and $z_0 \in \mathbb{C}$ and consider the power series:*

$$P(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \tag{1.12}$$

Then the usual convergence tests (such as ratio test or M-test) apply. Let

$$r := sup(|z - z_0| : \sum_{m=0}^{\infty} a_m (z - z_0)^m converges) \in \mathbb{R}_{\geq 0} \cup \{\infty\} \tag{1.13}$$

denote the radius of convergence of P(z). Then $\sum_{m=0}^{\infty} a_m(z - z_0)^m$ *converges absolutely and uniformly on* $D_{r'}(z_0) := \{z \in \mathbb{C} | \, |z - z_0| < r'\}$ *for any* $r' < r$ *and diverges for any* $z \in \mathbb{C}$ *such that* $|z - z_0| > r$*. The limit function P(z) is holomorphic on* $D_r(z_0)$ *and its derivative is:*

$$P'(z) = \sum_{m=0}^{\infty} m a_m (z - z_0)^{m-1}, \tag{1.14}$$

where the new power series converges on $D_r(z_0)$ *(i.e. it has the same radius of convergence as the original power series) and its limit is unique, (*$D_r(z_0)$ *is the open disk of radius* r *centered at* z_0*).*

Proof. Consult reference [1] on page 35 (Prop. 2.5). □

Proposition 1.1. *A function defined on an open subset* $S \subseteq \mathbb{C}$ *is analytic (holomorphic), if and only if it is locally given by a convergent power series.*

This is in contrast with real functions. It is important to note that the radius of convergence remains the same if we differentiate the power series even when the functions are defined over the real line. However, not every differentiable real function can be represented by a power series. A very famous example of a such function is:

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ e^{-1/x^2}, & \text{otherwise.} \end{cases} \tag{1.15}$$

Therefore, the notions of analyticity (locally given by convergent power series) and differentiability are in general different for real functions.

Let us introduce some important notions of Point-Set Topology.

Definition 1.3. *Let* X *be a set. A family* $\mathcal{T}$ *of subsets of* X *is called a topology on* X *if it satisfies the following conditions:*
(1) $X, \emptyset \in \mathcal{T}$
(2) the union of any collection of sets in $\mathcal{T}$ *is in* $\mathcal{T}$
(3) the intersection of finitely many sets in $\mathcal{T}$ *is in* $\mathcal{T}$

If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a topological space. The elements of $\mathcal{T}$ are called open sets of $(X, \mathcal{T})$. In addition to that, **any open set containing a point** $x \in X$ **is said to be an open neighbourhood of** x. [8]

Lemma 1.2. *A subset* S *of* X *is open, if and only if for every* $x \in S$ *there is an open set* S_x*, such that* $x \in S_x \subseteq S$*.*

Proof. " $\Rightarrow$ " If S is open, then choose $S_x = S$ for every $x \in S$.
" $\Leftarrow$ " Conversely, suppose that every point $x \in S$ lies within an open subset of S. Let us try to show that:

$$S = \bigcup_{x \in S_x, S_x \subseteq S} S_x \tag{1.16}$$

Let $y \in S$ be an arbitrary point in S. Then, by assumption, there exists an open set $S_y \subseteq S$ containing y. Since $\bigcup_{x \in S_x, S_x \subseteq S} S_x$ is the union of all such open sets, $S \subseteq \bigcup_{x \in S_x, S_x \subseteq S} S_x$ as this holds for all $y \in S$. To prove the opposite inclusion, note that $S_x \subseteq S,\ \forall x \in S$. Hence, so is the union in (1.16). This proves (1.16) and S is open as a union of open sets by Def. 1.3. $\square$

Definition 1.4. *Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subseteq X$ is said to be a basis of topology $\mathcal{T}$ if every element of $\mathcal{T}$ is a union of open sets from $\mathcal{B}$.*

Let (X, d) be a metric space. Then, let $\mathcal{B}_m$ be the family of open metric balls in X, i.e. all sets $B_r(x) = \{y \in X |\ d(y, x) < r\}$, such that $x \in X$ and $r > 0$. We have the following definition:

Definition 1.5. *The metric topology on space X is the topology created from the basis $\mathcal{B}_m$ and is denoted by $\mathcal{T}_m$. In this case we say that the metric d induces the topology $\mathcal{T}_m$ on X.*

It can be shown that the metric topology $\mathcal{T}_m$ is indeed a topology in the sense of definition 1.3 (see references).

If we are talking about analytic functions, we need an open neighbourhood (an open set in the topological sense) over which the power series to converge. Prop. 1.2 then guarantees us that any analytic function will be differentiable not only at the point around which the power series is centered, but also on the whole open neighbourhood. However, there exist functions which are only differentiable at a single point. This means that they will not possess power series anywhere because a single point cannot represent an open neighbourhood in the metric topology induced by the absolute value metric.[2] The reason why a single point is not an open neighbourhood is because every open ball around this point will also contain elements different from the point, i.e. there does not exist an open set around the point contained in the set defined by the point itself.[3]

Example 1.3. *Let $f : \mathbb{C} \to \mathbb{C}, z \mapsto z^2|z|$.*

The function is continuous everywhere, but is not differentiable on $\mathbb{C}\backslash\{0\}$. This can be checked using the Cauchy-Riemann equations for example. The only point that is left to check differentiability is the origin. From the definition of a derivative, we have:

$$f^{'}(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{z^2|z| - 0}{z - 0} = \lim_{z \to 0} z|z| = 0 \quad (by\ arithmetic\ of\ limits) \tag{1.17}$$

Hence f is differentiable at $z = 0$, but we cannot say if it is more than once differentiable there for obvious reasons. Since this happens at only one point, there does not exist a power series that will converge to $f(z)$ for any $z \in \mathbb{C}$ and therefore it is nowhere analytic.

1.2 Classification of singular points of homogeneous linear equations

As mentioned earlier, often we cannot solve differential equations in closed form. However, the theory of linear differential equations can usually help us predict the behaviour of the solutions (if

[2] Which is precisely the topology used for defining the convergence on an open disk in Thm. 1.2.
[3] Such sets consisting of only one point are called singletons.

they exist) near a point z_0 ,where we suspect that the equation is well behaved (for example does not blow up), without actually knowing the solutions or even how to find them. It suffices to explore the coefficient functions of the differential equation in a neighbourhood of z_0. A homogeneous linear differential equation of order n can be written in the form:

$$y^{(n)}(z) + q_{n-1}(z)y^{(n-1)}(z) + \cdots + q_1(z)y^{(1)}(z) + q_0(z)y(z) = 0, \tag{1.18}$$

where $y^{(k)}(z) = d^k y/dz^k$. In what follows we assume that the coefficient functions $q_0(z)$,...,$q_{n-1}(z)$ have been defined for complex as well as for real arguments. To continue our local analysis of (1.18), we start by classifying a point z_0 as an ordinary point, a regular singular point, or an irregular singular point of a homogeneous linear equation.

Definition 1.6. *The point z_0 ($z_0 \neq \infty$) is called an ordinary point of (1.18) if the coefficient functions $q_0(z)$,...,$q_{n-1}(z)$ are all analytic in a neighbourhood of z_0 in the complex plane. [6]*

Example 1.4.

$$y^{''''} = sin(z)y. \tag{1.19}$$

Every point $z_0 \neq \infty$ is an ordinary point of (1.18) because $sin(z)$ is differentiable everywhere except when $|z| \to \infty$, where it has an essential singularity.

Example 1.5.

$$y^{'} = z^2|z|y. \tag{1.20}$$

There are no ordinary points in the complex plane because $z^2|z|$ is nowhere analytic as discussed above.

It can be proved that all n linearly independent solutions of (1.18) are locally analytic in neighbourhoods containing ordinary points. Moreover, as any solution can be expanded in a Taylor series about an ordinary point z_0, $y(z) = \sum_{m=0}^{\infty} a_m(z - z_0)^m$, then the radius of convergence of this series is at least as large as the distance to the nearest singularity of the coefficient functions in the complex plane:

Theorem 1.3. *The location of any singularities that the solutions could have must coincide with the location of the singularities of the coefficient functions. The solutions of a linear equation cannot have singularities at any other points. [6]*

Definition 1.7. *The point z_0 ($z_0 \neq \infty$) is called a regular singular point of (1.18) if not all of $q_0(z)$,...,$q_{n-1}(z)$ are analytic, but if all of $(z - z_0)^n q_0(z)$, $(z - z_0)^{n-1} q_1(z)$, ..., $(z - z_0)q_{n-1}(z)$ are analytic in a neighborhood of z_0. [6]*

Example 1.6.

$$sin(z-3)y^{'''} = y \tag{1.21}$$

There is a regular singular point at $z = 3$ as $(z-3)^3/sin(z-3)$ has a removable singularity at $z = 3$ and hence, by redefining it at this point, we obtain an analytic function.

Definition 1.8. *The point z_0 ($z_0 \neq \infty$) is called an irregular singular point of (1.18) if it is neither an ordinary point, nor a regular singular point.*

Typically, at an irregular singular point, all solutions exhibit an essential singularity.

1.3 Local behaviour near ordinary points of homogeneous linear equations

In this section we explore the local behaviour of solutions to differential equations which cannot be expressed in closed form. The general procedure consists of first classifying the point z_0 as an ordinary, a regular singular, or an irregular singular point and then selecting a suitable form for the series based on this classification. A local series expansion about z_0 of a solution to a differential equation is an example of a perturbation series. A perturbation series is a series in powers of a small parameter. Here the small parameter is the distance between z and z_0. If $y(z)$ is analytic near z_0, then z_0 is an ordinary point and the perturbation expansion is a Taylor series in powers of $(z - z_0)$. The n^{th} approximant (the sum of the first n terms of the perturbation expansion) to the local behavior near an ordinary point becomes a more accurate approximation to the solution as $|z - z_0|$ becomes smaller, or as n increases, or both.

Example 1.7. *Local analysis of the Airy equation at 0.*

We now restrict to the case of real functions so instead of z as a variable, we use x to avoid confusion. The Airy equation is given by:

$$y^{''} = xy. \tag{1.22}$$

Now, using Def. 1.6 and Thm. 1.3, we can conclude that the solutions to the equation near the point $x = 0$ are analytic.[4] This implies that there exists a convergent power series centered at $x = 0$, such that $y(x) = \sum_{m=0}^{\infty} a_m x^m$. The radius of convergence is infinite, since x is differentiable everywhere. On the other hand, Thm. 1.2 implies that we can differentiate the power series term by term and the new power series will converge to $y^{'}(x)$. The same holds for the second derivative and, by plugging in the power series into the Airy equation, we obtain:

$$\sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} = \sum_{m=0}^{\infty} a_m x^{m+1} \tag{1.23}$$

By shifting the indices, the RHS becomes: $\sum_{m=3}^{\infty} a_{m-3}x^{m-2}$. Again, by Thm. 1.2, we know that the limit to which the power series converge is unique. However, this still does not give us right to equate the coefficients before the different powers of x, which is what we are looking for.

Proposition 1.2. *Let $P(z) = \sum_{m=0}^{\infty} a_m(z-z_0)^m$ be as in Thm. 1.2 and has a radius of convergence $r > 0$. Then for $m \geq 0$, a_m are given by the formula below and they are unique:*

$$a_m = \frac{P^{(m)}(z_0)}{m!} \tag{1.24}$$

[4]Note that this is not quite true, since Def. 1.6 applies to functions defined over the complex plane. In our case, for an ordinary point, we require only the coefficient function before the highest derivative to be different from 0 at this point. For the Airy equation, the coefficient function before $y^{''}$ is 1 and hence our initial assumption holds.

Proof. From Thm. 1.2 we know that we can differentiate the power series term by term and the new power series will have the same radius of convergence r and converge to $P^{'}(z)$. Hence, by induction on m, we conclude that this holds for all $m \geq 0$. In particular, the k^{th} derivative of the power series will converge to:

$$P^{(k)}(z) = \sum_{m=k}^{\infty} k(k-1)...(m-k+1)a_m(z-z_0)^{m-k} \tag{1.25}$$

By a straightforward evaluation, we get $a_0 = P(z_0) = P^{(0)}(z_0)$. Using (1.25), we get $P^{(k)}(z_0) = k!a_k$, which proves (1.24). □

We now consider the following lemma which will be useful when considering finite sums:

Lemma 1.3. *Let $S \subseteq \mathbb{C}$ be a non-empty open set. Let $i \in \mathbb{N} \cup \{0\}$ and t^i denote the polynomial function $S \to \mathbb{C}, z \mapsto z^i$ (we regard the polynomial functions as part of the vector space of $\mathbb{C}^S$ (function space) over the field of complex numbers). Then $1, t, t^2, ..., t^n$ are linearly independent in $\mathbb{C}^S$.*

Proof. Let $a_0, a_1, ..., a_n \in \mathbb{C}$, such that $a_0 + a_1 t + ... + a_n t^n = \underline{0}$, where $\underline{0}$ is the zero vector in the vector space $\mathbb{C}^S$. This implies: $a_0 + a_1 z + ... + a_n z^n = 0$ for all $z \in S$. However, using a corollary of the fundamental theorem of algebra we know that every polynomial of degree n has exactly n zeros. The above equation, on the other hand, must hold for all $z \in S$ which are "more" than n, since any non-empty open set in $\mathbb{C}$ contains infinitely many elements[5]. Therefore we can conclude that $a_0 = a_1 = ... = a_n = 0$ as required. □

Let us come back to the Airy equation. We ended up with:

$$\sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} = \sum_{m=3}^{\infty} a_{m-3}x^{m-2} \tag{1.26}$$

Equating the coefficients of x^{m-2} in $y^{''}$ and xy (by Prop. 1.2) gives:

$$a_m m(m-1) = 0 \; for \; m = 0, 1, 2, \tag{1.27}$$

$$a_m m(m-1) = a_{m-3} \; for \; m = 3, 4, ..., \tag{1.28}$$

which can be solved in closed form only for $m = 0, 1, 2$. The first equation is already satisfied for $m = 0$ and $m = 1$. Hence, a_0 and a_1 are arbitrary constants. Also, $a_2 = 0$. The solutions of the second equation are:

$$a_{3m} = \frac{a_0 \Gamma(2/3)}{3^m m! 3^m \Gamma(m + 2/3)}, \tag{1.29}$$

[5]Again, this is not entirely correct. If we equip $\mathbb{C}$ with the discrete topology (which is defined such as all subsets of $\mathbb{C}$ are open - it is the largest possible topology), then even single points are open. However, in all the theory above, we have always considered $\mathbb{C}$ equipped with the standard metric and in all such cases, the claim holds.

$$a_{3m+1} = \frac{a_1 \Gamma(4/3)}{3^m m! 3^m \Gamma(m+4/3)}, \tag{1.30}$$

$$a_{3m+2} = 0, \tag{1.31}$$

where $\Gamma(z)$ is the gamma function. If we now define $c_1 = a_0 \Gamma(2/3)$ and $c_2 = a_1 \Gamma(4/3)$, then the general solution of the Airy equation is:

$$y(x) = c_1 \sum_{m=0}^{\infty} \frac{x^{3m}}{9^m m! \Gamma(m+2/3)} + c_2 \sum_{m=0}^{\infty} \frac{x^{3m+1}}{9^m m! \Gamma(m+4/3)}. \tag{1.32}$$

Let us prove that the two series in equation (1.31) are linearly independent. We want to use the Wronskian and for this purpose define the two series as different functions:

$$h(x) = \sum_{m=0}^{\infty} \frac{x^{3m}}{9^m m! \Gamma(m+2/3)}; \quad g(x) = \sum_{m=0}^{\infty} \frac{x^{3m+1}}{9^m m! \Gamma(m+4/3)}. \tag{1.33}$$

In order for the Wronskian test to hold, we also need a closed interval containing the point $x = 0$ (as this the point around which we are expanding the series). Let $a, b \in \mathbb{R}$ be such that $0 \in [a, b]$.[6] The easiest point to evaluate the Wronskian is at $x = 0$. After easy manipulations we obtain:

$$h(0) = \frac{1}{\Gamma(2/3)}; \quad g(0) = 0; \quad h^{'}(0) = 0; \quad g^{'}(0) = \frac{1}{\Gamma(4/3)}. \tag{1.34}$$

Therefore:

$$W[h,g](0) = \begin{vmatrix} h(0) & g(0) \\ h^{'}(0) & g^{'}(0) \end{vmatrix} = h(0)g^{'}(0) - g(0)h^{'}(0) = \frac{1}{\Gamma(2/3)\Gamma(4/3)} \neq 0 \tag{1.35}$$

Hence, the two series are linearly independent on any closed interval in $\mathbb{R}$ containing 0. We obtained two linearly independent solutions, each multiplied by a constant of integration, even though we started with only one series! The general solution is approximated for all finite x in terms of c_1 and c_2 by a rapidly convergent Taylor expansion. As $|x|$ gets smaller, the expansion converges faster; fewer terms are required for any desired accuracy. The radii of convergence of these series are infinite because the Airy equation has no singular points. Note, however, that c_1 and c_2 must be known before $y(x)$ can be approximated.

It is conventional to define two specific solutions of the Airy equation by [6]:

$$Ai(x) := 3^{-2/3} \sum_{m=0}^{\infty} \frac{x^{3m}}{9^m m! \Gamma(m+2/3)} - 3^{-4/3} \sum_{m=0}^{\infty} \frac{x^{3m+1}}{9^m m! \Gamma(m+4/3)}, \tag{1.36}$$

$$Bi(x) := 3^{-1/6} \sum_{m=0}^{\infty} \frac{x^{3m}}{9^m m! \Gamma(m+2/3)} - 3^{-5/6} \sum_{m=0}^{\infty} \frac{x^{3m+1}}{9^m m! \Gamma(m+4/3)}. \tag{1.37}$$

$Ai(x)$ and $Bi(x)$ are called Airy functions. When x is not too large, the Taylor series in (1.36) and (1.37) give good approximations to the Airy functions. These functions are of primary importance in perturbative analysis of differential equations. In (1.36) and (1.37) c_1 and c_2 are chosen so that $Ai(x)$ is exponentially decreasing as $x \to +\infty$ and that $Bi(x)$ oscillates 90^o out of phase from $Ai(x)$ as $x \to -\infty$. It is not trivial to derive these properties from the series (1.36) and (1.37).

[6]Note that if h, g were not analytic, we cannot apply the Wronskian test, as Peano observed in 1889.

1.4 Local series expansions about regular singular points of homogeneous linear equations

In the previous section we saw that Taylor series are a good way to represent local behaviour of solutions to a differential equation near ordinary points. However, this procedure will break down if we apply it to regular singular points. It can be shown that if x_0 is a regular singular point of (1.18), at least one of the solutions of (1.18) will be of the form:

$$y(z) = (z - z_0)^{\alpha} H(z), \tag{1.38}$$

where H(z) is analytic at z_0. The RHS of (1.38) is called **Frobenius series** and the number $\alpha(\alpha \neq 0)$ is called an indicial exponent. The Frobenius method allows us to find the solutions of homogeneous linear differential equations, but we are not going to discuss it here. Consult reference [6] for more information.

2 Asymptotic Series

This section starts with introducing the $O - o$ symbols. We first note that when dealing with functions, the independent variable can be real or complex. More generally, it will belong to a so-called *Hausdorff* space:

Definition 2.1. *A topological space $(X, \mathcal{T})$ (as in Def. 1.3) is* ***Hausdorff*** *if any two distinct points in X have disjoint neighbourhoods. In other words, if $x, y \in X$ and $x \neq y$, then there exist $\mathcal{U}, \mathcal{V} \in \mathcal{T}$, such that $x \in \mathcal{U}$, $y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.*

The usefulness of the *Hausdorff* condition is that it allows us to distinguish two different points in a given space. There are many spaces that are not *Hausdorff*. For example, let a set X be non-empty and such that X has at least two distinct points in it. If X is equipped with the **indiscrete topology** $\{\emptyset, X\}$, then the resulting space is not *Hausdorff*. This topology consists of only two elements, the empty set and X itself and one can easily show that it satisfies Def. 1.3. It is also the "smallest" topology, while the discrete one is the "largest" possible. Mathematically, we should say that the indiscrete topology is **coarser** (i.e. contains fewer open sets) than the discrete topology. Indeed, if $x, y \in X, x \neq y$, then any open neighbourhood containing either of these points has to be non-empty and the only such set in this topological space is X, but it is not disjoint from itself. While the indiscrete topology is slightly more abstract[7], the following proposition will give us an infinite number of useful examples:

Proposition 2.1. *Any metric space (X, d) endowed with the metric topology $\mathcal{T}_m$ (as in Def. 1.5), is Hausdorff.*

Proof. Consult reference [8]. □

[7] The abstraction comes from the fact this topology is not induced by a metric, as one can easily show.

In particular, the "hidden" metric topology induced by the absolute value metric in Thm. 1.2 satisfies the *Hausdorff* condition. The notion of a *Hausdorff* space is introduced here only for completeness and for more information refer to [3]. We are now going to focus on the applications of these concepts with simple examples.

2.1 The O-symbol

Definition 2.2. *Let $(X, \mathcal{T})$ be a Hausdorff topological space and $S \subseteq X$. Then for any real- or complex-valued functions of x defined when $x \in S$ (for example φ, ζ), we write $\varphi = O(\zeta)$ in S if there exists a positive constant A, independent of x, such that $|\varphi| \leq A|\zeta|$ for all $x \in S$. We also write $\varphi = O(\zeta)$ as $x \to x_0$ if there exists a positive constant A and an open neighbourhood $U \subseteq X$ of x_0, such that $|\varphi| \leq A|\zeta|$ for all $x \in \mathcal{T}_U$, where $\mathcal{T}_U$ is the subspace (induced) topology (i.e. $\mathcal{T}_U = \{U \cap S | \ S \in \mathcal{T}\}$). [3]*

The definition is taken from [3], but is rather abstract and confusing. In order to explain it, we first note the "abuse" of the equality sign. Let $x \in S$ be arbitrary. If:

$$\varphi = O(\zeta) \tag{2.1}$$

holds, then obviously this implies that $3\varphi = O(\zeta)$. The reason for this is that the constant A, though independent of the variable x, is not unique (i.e. we can always choose a bigger constant, such that equation (2.1) and $3|\varphi(x)| \leq A^*|\zeta(x)|$ are both satisfied, where A^* is the new modified constant). This would imply that $3\varphi(x) = \varphi(x)$ for all $x \in S$, since x was arbitrary. This is true only for the constant 0 function, which will make the definition of the O symbol useless. To illustrate what is meant by the equality sign here, we provide the following instructive examples.

Imagine that instead of using the "less than" sign "$<$", we define a new operation "$= L$" with the same meaning. Then one can write: $4 = L(7)$, as 4 is less than 7. We can extend this to functions as well: $\sin(x) = L(x)$, for all x satisfying $0 = L(x)$. Continuing in this way, we can write: $4 = L(7)$, so $6 - 2 = L(7)$ and hence $6 = 2 + L(7)$, or even something in the form: $L(2) + L(9) = L(16)$. The last "equality" suggests that the sum of an element less than 2 and another element less than 9, is less than 16, which is true. The O symbol is used in the same manner [2]. Now consider the "equation":

Example 2.1.

$$O(\ln(x)) + O(\ln(x^n)) = O(\ln(x)) \ when \ x \to \infty, \ n \in \mathbb{R} \ and \ 0 \leq n \leq N, \tag{2.2}$$

where N is a fixed finite number. We are clearly considering the topological space $(\mathbb{R}, \mathcal{T}_m)$ induced by the d_1[8] metric, and hence it is automatically *Hausdorff* by Prop. 2.1. The reader might also notice that we have put the independent variable in (2.2), but not in (2.1). It is a matter of convention

[8]The d_1 metric used in the first section was over the set of continuous functions. When the set is just the real numbers, the d_1 metric corresponds to the absolute value one.

and in certain cases we include the independent variable for the sake of clarity. Now, (2.2) means that for any pair of functions h, g, such that:

$$h(x) = O(\ln(x))\ (x \to \infty), \quad g(x) = O(\ln(x^n))\ (x \to \infty), \tag{2.3}$$

we have:

$$h(x) + g(x) = O(\ln(x)) \tag{2.4}$$

Equation[9] (2.2) holds for all n in the closed interval $[0, N]$. The example is trivial as we simply need to use properties of the real logarithmic function to bring n before it in order to prove it, but it is important to realize that we need to take arbitrary functions that satisfy (2.3) and show that their sum satisfies (2.4). Hence, we can generalize (2.4) to (2.2). Some other simple examples could be:

$$O(x^2) + O(x^7) = O(x^7), \quad (x \to \infty), \tag{2.5}$$

$$e^{O(N)} = O(M), \quad (-\infty < x < \infty),\ with\ N, M \in \mathbb{R}_{>0}, \tag{2.6}$$

$$e^{O(x)} = e^{O(x^2)}, \quad (x \to \infty). \tag{2.7}$$

Equations (2.5) and (2.7) are trivial. (2.6) it might look strange as the independent variable is missing. The explanation is given at the end of this section. We shall now discuss a few simple concepts.

Let φ and ζ be as in definition 2.2 and assume that $\varphi = O(\zeta)$ is true, but $\zeta = O(\varphi)$ is false. If there is a third function f, defined over the same topological space, satisfying:

$$f = O(\varphi), \tag{2.8}$$

then clearly it also satisfies:

$$f = O(\zeta). \tag{2.9}$$

We say that equation (2.8) is a **refinement** of (2.9). An equation of the form (2.8) is called the best refinement if it cannot be refined, i.e. there exist positive constants a and A such that $a|\varphi(x)| \leq |f(x)| \leq A|\varphi(x)|$. We suggest the following example:

Example 2.2.

$$Nx + Mx\cos(\ln(x)) = O(x), \quad (x \to \infty)\ with\ N, M \in \mathbb{R}_{>0}\ and\ N > M. \tag{2.10}$$

We cannot refine (2.10) as the LHS of it lies between $(N - M)x$ and $(N + M)x$. In some cases the best refinement might not exist. Consider for $n \in \mathbb{N}$:

$$e^{-x} = O(x^{-n}), \quad (x \to \infty). \tag{2.11}$$

(2.11) is obviously true, but for no value of n is the best refinement, since $e^{-x} = O(x^{-n-1})$ is always a better one. We now focus on the notion of **uniformity**, which in some ways resembles uniform convergence as we know from Analysis. Let us start once again with an example.

[9] I drop the quotation marks, which refer to something said in a sloppy way, but the reader should keep in mind that we are not actually dealing with equations.

Example 2.3. *Consider n arbitrary bounded functions ζ_i, $i = 1, 2, .., n$ as before defined over a subset S of a Hausdorff topological space, where $n \in \mathbb{N}$, and let l be a positive real number. Then, we have:*

$$\Big(\sum_{i=1}^{n} \zeta_i(x)\Big)^l = \sum_{i=1}^{n} O(\zeta_i^l(x)), \quad \forall x \in S. \tag{2.12}$$

Proof. To prove this, try to bound the modulus of the LHS of (2.12). For example,

$$\Big|\Big(\sum_{i=1}^{n} \zeta_i(x)\Big)^l\Big| \leq \Big(\sum_{i=1}^{n} |\zeta_i(x)|\Big)^l \leq [n \max\left(|\zeta_1(x)|, |\zeta_2(x)|, ..., |\zeta_n(x)|\right)]^l \tag{2.13}$$

$$\leq n^l \max\left(|\zeta_1(x)|^l, |\zeta_2(x)|^l, ..., |\zeta_n(x)|^l\right) \leq n^l \Big(\sum_{i=1}^{n} |\zeta_i(x)|^l\Big). \tag{2.14}$$

(2.13) and (2.14) suggest that we have found constants $A_1, A_2, ..., A_n$ such that:

$$\Big|\Big(\sum_{i=1}^{n} \zeta_i(x)\Big)^l\Big| \leq \sum_{i=1}^{n} A_i |\zeta_i(x)|^l, \quad \forall x \in S, \tag{2.15}$$

where we stress that the constants depend on l (in particular we can set $A_i = n^l$, $\forall i = 1, 2, ..., n$) and hence (2.12) is not uniform with respect to l or at least that we have not proved the existence of constants independent of l. □

However, let us take a look at the following example:

Example 2.4.

$$\Big(\frac{l}{x^2 + l^2}\Big)^l = O\Big(\frac{1}{x^l}\Big), \quad (1 < x < \infty),\ l > 0. \tag{2.16}$$

We claim that the constant involved in this example can be chosen independently of l. To see this, note that $x^2 + l^2 = (x - l)^2 + 2lx \geq 2kl$ and hence:

$$\Big(\frac{l}{x^2 + l^2}\Big)^l \leq \frac{1}{(2x)^l}. \tag{2.17}$$

Since $2^{-l} < 1, \forall l > 0$, then we can choose the constant in (2.16) to be 1, such that the expression holds uniformly with respect to l. We can also take a different point of view and consider the function on the LHS of (2.16) as a function of two variables, namely x and l. Then, if we plot the LHS of (2.16) over a subset of the $x - l$ plane, i.e. $1 < x < \infty,\ l > 0$, we can bound its height by a constant multiple of the function $1/x^l$. We proceed with some basic, but important properties of the O symbol.

Proposition 2.2. *Let $\{\varphi_i |\ i = 1, 2, ...\}$ and $\{\zeta_i |\ i = 1, 2, ...\}$ be finite sets (with the same number of elements) of functions as in Def. 2.2, such that $\varphi_i = O(\zeta_i)$ for all i. Then, for any set of constants a_i, the following holds:*

$$\sum_i a_i \varphi_i = O(\sum_i a_i \zeta_i) \tag{2.18}$$

The proof is given in reference [3]. However, we have to be careful, as in some cases the value to which the independent variable tends to is crucial. For instance, $x^2 = O(x)$ when $x \to 0$ is true, but clearly fails when $x \to \infty$, as one can easily show that x^2 is not Lipschitz for all x.

Proposition 2.3. *Let φ_i, ζ_i be the same as in Prop. 2.2. Then, we have:*

$$\prod \varphi_i = O(\prod \zeta_i) \tag{2.19}$$

Proof. Let $x \in S$ (where S is as in Def. 2.2) be arbitrary. Then, by assumption, for every i there exists a positive constant A_i, such that $|\varphi_i(x)| \leq A_i|\zeta_i(x)|$. Therefore, we have:

$$\prod |\varphi_i(x)| \leq \prod A_i|\zeta_i(x)| = \prod A_i \prod |\zeta_i(x)| = A \prod |\zeta_i(x)|, \tag{2.20}$$

where $A := \prod A_i$ is finite (as $A_i < \infty$ for all i). Since x was arbitrary, (2.20) holds for all $x \in S$, hence proving (2.19). □

At this stage the reader should notice something peculiar about our proofs. Sometimes, we only take an arbitrary x at the start and try to prove our claim for all x in a given set, but in other cases we take an arbitrary function instead. The reason for this is because the O symbol actually refers to classes of functions rather than the individual functions themselves. For example, when we write: $O(\ln(x)) + O(\ln(x^n)) = O(\ln(x))$ *when* $x \to \infty$, we mean that the class $O(\ln(x)) + O(\ln(x^n))$ is contained in the class $O(\ln(x))$. That is why we need to choose arbitrary members of the class on the LHS of (2.2) and we did not need an arbitrary x as we assumed that $x \to \infty$ (i.e. we sort of know about which $x's$ we are talking about). This does not mean that we cannot refer to a single function when using the O symbol. In Props. 2.2 and 2.3, we had a condition for each of φ_i rather than a condition for classes of functions. On the other hand, equation (2.6) clearly refers to classes of functions. In general, it should be clear from the example whether we deal with classes or single functions.

2.2 The o-symbol

Definition 2.3. *Let (X, d) be a metric space and S, x, φ and ζ be as in Def. 2.2. Then, we write:*

$$\varphi = o(\zeta) \quad as\ x \to x_0, \tag{2.21}$$

if for every $\epsilon > 0$, there exists an open neighbourhood $U_\epsilon \subseteq X$ of x_0 (i.e. an open ball with radius ϵ around x_0), such that $|\varphi| \leq \epsilon|\zeta|$ for all $x \in \mathcal{T}_{U_\epsilon}$.

Note that we do not actually require x_0 to belong to S. The definition merely states the existence of an open neighbourhood around x_0 that has a non-empty intersection with S. Before providing an example, we define what the closure of a set means:

Definition 2.4. *A point x in a topological space $(X, \mathcal{T})$ is called an **adherent point** of a subset $S \subset X$, if every open neighbourhood $U \subseteq X$, containing x, has a non-empty intersection with S. The set of all adherent points of S is called the **closure** of S and is denoted by $\overline{S}$.*

For example, $\overline{(0, 1/\sqrt{5})} = [0, 1/\sqrt{5}]$, but neither 0, nor $1/\sqrt{5}$ belong to $(0, 1/\sqrt{5})$. In particular, if we take $x_0 = 0$ we can construct a sequence of points $x_n \in (0, 1/\sqrt{5})$ that lie within open balls of radius $1/n,\ n \in \mathbb{N}$ (i.e. $x_n \in B_{1/n}(0)$) and approach 0. In this case the metric space is not complete as the sequence converges in $\mathbb{R}$ to 0 (hence, it is Cauchy), but does not converge in $(0, 1/\sqrt{5})$. It is important to note that we do not require $n \geq 3$ as the reader might think, since $\sqrt{5} \approx 2.24$, because there is nothing in the definition to prevent this intersection to be the whole set S (in our case $S = (0, 1/\sqrt{5})$). Therefore, we can still write something of the form:[10]

$$e^{-\frac{1}{x}} = o(x), \quad as\ x \to 0_+{}^{11}, \tag{2.22}$$

even if we exclude the zero in our space. To decode the abstraction above, read the **o-symbol** as **"something that tends to zero, multiplied by"**. In practice, if we write $\varphi = o(\zeta),\ as\ x \to x_0$, we mean the following:

$$\lim_{x \to x_0} \frac{\varphi(x)}{\zeta(x)} = 0. \tag{2.23}$$

To see how this works with our example, compute a limit as in (2.23):

$$\lim_{x \to 0_+} \frac{e^{-\frac{1}{x}}}{x}. \tag{2.24}$$

Note that, if we let x tend to 0 from both sides, the limit will not exist. A standard trick at this stage is to make a change of variables, so let $u := 1/x$ and hence $u \to \infty$ as $x \to 0_+$. We are now in a position to apply L'Hôpital's rule, since, no matter how big u is, we can always find an open neighbourhood containing u where the limit exists, which was not possible in (2.24) around the zero. Therefore:

$$\lim_{x \to 0_+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{u \to \infty} u e^{-u} = \lim_{u \to \infty} \frac{u}{e^u} = 0, \tag{2.25}$$

as required. In asymptotic analysis, the $o - symbol$ is less useful than the $O - symbol$, since it suppresses a lot more information. We are usually interested in how fast something approaches zero rather than simply that it goes to zero.

2.3 Asymptotic series

Often, we have a situation where a given function φ, as $x \to \infty$, has an infinite sequence of $O - formulas$, such that each $n+1$ formula is an improvement upon the n^{th} one. Let us assume that the sequence is of the following type [2]:

Consider the sequence of linearly independent functions $(\zeta_0, \zeta_1, \zeta_2 ...)$ satisfying:

$$\zeta_1 = o(\zeta_0), \quad \zeta_2 = o(\zeta_1), ... \quad as\ x \to \infty, \tag{2.26}$$

[10] It is not surprising that the o symbol refers to classes of functions (as the big O), but in this example we are talking about specific members of the class, i.e. $e^{-\frac{1}{x}}$ and x.

[11] Formally, I should first define a function $t : \mathbb{R} \to \mathbb{R};\ x \mapsto x$ and write $o(t)$, rather than $o(x)$. The same holds for the whole chapters 2 and 3, but as we said earlier, it is a matter of convention and clarity.

and there is a sequence of constants $(a_0, a_1, a_2, ...)$, such that we have the following $O-formulas$:

$$\begin{cases} \varphi = O(\zeta_0) \quad as\ x \to \infty \\ \varphi(x) = a_0\zeta_0(x) + O(\zeta_1) \quad as\ x \to \infty \\ \varphi(x) = a_0\zeta_0(x) + a_1\zeta_1(x) + O(\zeta_2) \quad as\ x \to \infty \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi(x) = a_0\zeta_0(x) + a_1\zeta_1(x) + ... + a_{n-1}\zeta_{n-1}(x) + O(\zeta_n) \quad as\ x \to \infty \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{2.27}$$

Now, we can ask ourselves if each consecutive formula is an improvement of the previous one. Take for example the first two. We want to show that $a_0\zeta_0(x) + O(\zeta_1) = O(\zeta_0)$ as $x \to \infty$.

Proof. We suggest the following proof. Let h be an arbitrary function satisfying[12] $h = O(\zeta_1)$ as $x \to \infty$. This means that there exists a positive constant A such that $h(x) \leq A\zeta_1(x)$ when $x \to \infty$. Now, consider the limit:

$$\lim_{x\to\infty} \frac{h(x)}{1\zeta_0(x)} \leq \lim_{x\to\infty} \frac{A\zeta_1(x)}{\zeta_0(x)} = A \lim_{x\to\infty} \frac{\zeta_1(x)}{\zeta_0(x)} = 0, \tag{2.28}$$

by arithmetics of limits and the assumption that $\zeta_1 = o(\zeta_0)\ as\ x \to \infty$. Since this holds for all such functions h, we can write: $O(\zeta_1) = o(1)\zeta_0(x)\ as\ x \to \infty$ (hence the 1 in the denominator of (2.28)). Therefore, we have:

$$a_0\zeta_0(x) + O(\zeta_1) = a_0\zeta_0(x) + o(1)\zeta_0(x) = (a_0 + o(1))\zeta_0(x) = O(\zeta_0) \quad as\ x \to \infty, \tag{2.29}$$

as required. □

Accordingly, the third formula improves the second one and so on. In order to represent the whole sequence (2.27) with a single formula, we use the following notation:

Definition 2.5. *An asymptotic series (expansion) of $\varphi(x)$ to N terms, is said to be the RHS of (2.30):*

$$\varphi(x) = \sum_{n=0}^{N} a_n\zeta_n(x) + O(\zeta_{N+1}), \quad x \to \infty. \tag{2.30}$$

Theorem 2.1. *The coefficients a_n in (2.30) are unique for a given $\varphi(x)$.*

Proof. We prove this by contradiction. Assume that there is another set of constants $\{b_0, b_1, ..., b_N\}$ such that $a_n \neq b_n$ for at least one $n \in \{0, 1, ..., N\}$ and:

$$\varphi(x) = \sum_{n=0}^{N} b_n\zeta_n(x) + O(\zeta_{N+1}), \quad x \to \infty. \tag{2.31}$$

[12] I omit saying that each function should be defined over a Hausdorff topological space, since almost all examples are trivially defined over the real line with the standard metric, but the reader should not lose track of where we started from.

Let m be the smallest number such that $a_m \neq b_m$. m could be zero, N or anything in-between. If we now subtract (2.31) from (2.30), we obtain:

$$0 = (a_m - b_m)\zeta_m(x) + O(\zeta_{m+1}), \quad x \to \infty. \tag{2.32}$$

Since $a_m \neq b_m$ by assumption, we can divide through $b_m - a_m$, which implies:

$$\zeta_m = O(\zeta_{m+1}). \tag{2.33}$$

This is a contradiction as $\zeta_{m+1} = o(\zeta_m)$ by assumption in (2.26). □

Sometimes, it could turn out that the coefficients a_n are all zero. We suggest the following limit:

$$\lim_{x\to\infty} \frac{e^{-x}}{\ln\left(1 + 1/x^n\right)}, \quad n \in \mathbb{N}. \tag{2.34}$$

Both functions in the numerator and denominator are well-defined and tend to zero as $x \to \infty$, so we can apply the L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{e^{-x}}{\ln\left(1 + 1/x^n\right)} = \lim_{x\to\infty} \frac{(1 + x^{-n})e^{-x}}{nx^{-n-1}} = \lim_{x\to\infty} \frac{e^{-x}}{nx^{-n-1}} + \lim_{x\to\infty} \frac{e^{-x}}{nx^{-1}} = 0, \tag{2.35}$$

which implies that $e^{-x} = O(\ln\left(1 + 1/x^n\right))$ as $x \to \infty$ for all $n \in \mathbb{N}$. Therefore:

$$e^{-x} = 0 \cdot \ln(2) + 0 \cdot \ln(1 + 1/x) + \cdots + 0 \cdot \ln(1 + 1/x^N) + O(\ln(1 + 1/x^{N+1})), \tag{2.36}$$

where we consider the expansion of the first N terms. It is interesting to note that convergence of power series, as we know them from real or complex analysis, is not the same as convergence of asymptotic series. The difference is that if we are given a power series, we form a mathematical statement for a given x (the independent variable) when $n \to \infty$ (the index over which we sum the terms). On the other hand, if we have an asymptotic series, we form a statement about a given n when $x \to \infty$ (for example). Also, even if the asymptotic series converges, it does not need to converge to the initial function. A counterexample is (2.36), as $e^{-x} \neq 0,\ \forall x \in \mathbb{R}$.

2.4 Operations on asymptotic series

We restrict our attention to asymptotic series of the form:

$$\sum_{n=0}^{N} a_n x^n + O(x^{N+1}), \quad as\ x \to 0. \tag{2.37}$$

The series (2.37) is called a **formal power series** [2]. Note that Lem. 1.3 justifies that (2.37) is an asymptotic series, because we required the summed functions to be linearly independent, and hence the above discussion holds for formal series as well. For these series, addition and multiplication are defined in the usual way. There also exists a unit element, denoted by I, which is obviously of the form: $1 + 0 \cdot x + 0 \cdot x^2 + \cdots$. Hence, **the set of all formal power series forms a commutative ring**, but for the purposes of this discussion we will not delve into commutative algebra. Instead, we prove some simple properties:

Theorem 2.2. *Suppose that there exists a formal power series representing a function $\varphi(x)$, i.e.:*

$$\varphi(x) = \sum_{n=0}^{N-1} a_n x^n + O(x^N), \quad as\ x \to 0, \tag{2.38}$$

and that $\int_0^x \varphi(u)du$ exists for all sufficiently small x. Then, we can integrate the asymptotic series term by term:

$$\int_0^x \varphi(u)du = \sum_{n=0}^{N-1} \frac{a_n}{n+1} x^{n+1} + O(x^{N+1}), \quad as\ x \to 0. \tag{2.39}$$

Proof. Let N be given and fixed. Then, by definition of the $O - symbol$, there exist positive constants A and a such that:

$$\Big|\varphi(u) - \sum_{n=0}^{N-1} a_n u^n\Big| \leq A|u|^N, \quad when\ |u| < a. \tag{2.40}$$

Now, if $|x| < a$, consider the modulus of the difference between the integral and the sum in (2.39) with $x \to 0_+$ (the case $x \to 0_-$ follows in a similar fashion):

$$\Big|\int_0^x \varphi(u)du - \sum_{n=0}^{N-1} \frac{a_n}{n+1} x^{n+1}\Big| = \Big|\int_0^x \varphi(u)du - \int_0^x \sum_{n=0}^{N-1} a_n u^n du\Big| = \tag{2.41}$$

$$\Big|\int_0^x \Big(\varphi(u) - \sum_{n=0}^{N-1} a_n u^n\Big)du\Big| \leq \int_0^x \Big|\varphi(u) - \sum_{n=0}^{N-1} a_n u^n\Big|du \leq \int_0^x A|u|^N du = \tag{2.42}$$

$$A\int_0^x u^N du = \frac{A}{N+1} x^{N+1} = \frac{A}{N+1}|x|^{N+1} = O(x^{N+1}). \tag{2.43}$$

In (2.41) we interchanged the integral and the sum, as the series consists of finitely many terms. In (2.42) the monotonicity of the integral and (2.40) were used. In (2.43) we used that x and u are bigger than zero by assumption. This proves the theorem. □

Theorem 2.3. *Assume that:*

$$\varphi(x) = \sum_{n=0}^{N} a_n x^n + O(x^{N+1}),\ (x \to 0); \quad \varphi'(x) = \sum_{n=0}^{N-1} b_n x^n + O(x^N),\ (x \to 0), \tag{2.44}$$

for some constants b_n. Then $b_n = (n+1)a_{n+1}$, as expected by termwise differentiation.

Proof. Assume again that $x \to 0_+$. Let $N \geq 1$ be fixed. Define a new function h_N, such that:

$$h_N(x) := \varphi(x) - \sum_{n=0}^{N-1} \frac{b_n}{n+1} x^{n+1}. \tag{2.45}$$

Now, since φ is differentiable and the sum has finitely many terms, then h_N is also differentiable $\Rightarrow h_N^{'}(x) = O(x^N)$ $(x \to 0)$[13]. We can now apply the Mean Value Theorem on the closed interval $[0, x]$. This implies that there exists a point $c \in (0, x)$, such that: $h_N(x) - h_N(0) = h_N^{'}(c)(x-0) = O(x^{N+1})$. Since N was arbitrary, we conclude that:

$$h_N(x) - h_N(0) = \varphi(x) - \sum_{n=0}^{N-1} \frac{b_n}{n+1} x^{n+1} - \varphi(0) = O(x^{N+1}),\ \forall N \geq 1 \Rightarrow \tag{2.46}$$

$$\varphi(x) = \varphi(0)(= a_0) + \sum_{n=0}^{N-1} \frac{b_n}{n+1} x^{n+1} + O(x^{N+1}). \tag{2.47}$$

The rest follows from relabelling the indices and Thm. 2.1, which assures uniqueness of the coefficients. □

Let us now focus for a bit on the differences between power series and asymptotic series:

Proposition 2.4. *Let P(z) be a convergent power series as in Thm. 1.2. Then it is asymptotic.*

Proof. Part of the proof is given in [5], but we extend it here because it is instructive. Let $P(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m$ be a convergent power series and let $\rho \in \mathbb{R}$, such that $\rho \in (r^{'}, r)$, where $r^{'}$ and r are as in Thm. 1.2. Since $P(z)$ converges, it is bounded, hence there exists a positive number $A^{'}$, such that $|a_m||z - z_0|^m < |a_m|\rho^m < A^{'},\ \forall m \in \mathbb{N} \cup \{0\}$ with the assumption that $|z - z_0| \leq r^{'}$. Therefore, excluding the first N terms of the infinite sum, we have:

$$\Big| \sum_{m=N}^{\infty} a_m (z - z_0)^m \Big| \leq \sum_{m=N}^{\infty} |a_m||z - z_0|^m = \sum_{m=N}^{\infty} |a_m| \rho^m \frac{|z - z_0|^m}{\rho^m} < A^{'} \sum_{m=N}^{\infty} \frac{|z - z_0|^m}{\rho^m} \tag{2.48}$$

$$= A^{'} \Big(\sum_{m=0}^{\infty} \frac{|z - z_0|^m}{\rho^m} - \sum_{m=0}^{N-1} \frac{|z - z_0|^m}{\rho^m} \Big) = A^{'} \Big(\frac{\rho}{\rho - |z - z_0|} - \frac{\rho(1 - \frac{|z-z_0|^N}{\rho^N})}{\rho - |z - z_0|} \Big) \tag{2.49}$$

$$= \frac{A' \rho^{1-N} |z - z_0|^N}{\rho - |z - z_0|} \leq \frac{A' \rho^{1-N}}{\rho - r^{'}} |z - z_0|^N = O(|z - z_0|^N), \tag{2.50}$$

where we used convergence of the geometric series, since $|z - z_0|/\rho < 1$ by assumption. Hence:

$$\sum_{m=0}^{\infty} a_m (z - z_0)^m = \sum_{m=0}^{N-1} a_m (z - z_0)^m + O(|z - z_0|^N), \tag{2.51}$$

as required. □

[13] To see this, take the sum on the LHS in the second equation of (2.44), one thus obtains $h_N^{'}(x)$.

Now, Thm. 1.2 says that we can differentiate the power series and the resultant series will also be convergent, and hence asymptotic as well. This is not true for asymptotic series. In Thm. 2.3 we also had to assume that derivative of the initial function exists and that we can form another asymptotic series to represent it. This is particularly apparent in the proofs for uniqueness of the coefficients for power series and asymptotic series. In Prop. 1.2 differentiation of the power series is justified, but not the case of asymptotic series, so we had to resort to a different technique in Thm. 2.1. Another key difference is that if we want to integrate a complex power series term by term, we need to make sure that it is (uniformly) convergent on an open neighbourhood in the complex plane, following Thm. 1.1. However, in Thm. 2.2 we did not impose such a condition. This is interesting as it says that the asymptotic series could be even divergent, but as long as the integral of the function it represents exists, we could still integrate it term by term, as essentially we are dealing with only finitely many terms of the series (as in the proof of Thm. 2.2). This means that asymptotic series are a lot more flexible than the power series, which are particularly rigid.

Up to know we have proved some properties of the asymptotic series, but we never had a formula of how to compute the coefficients a_n in (2.37). Let us first rewrite (2.37) in a different form. Since x tends to 0, we clearly have that $x^{N+1} = o(x^N),\ (x \to 0)$[14]. We want to prove that $O(x^{N+1}) = o(x^N),\ (x \to 0)$. Indeed, assume that h is a function that satisfies $h(x) = O(x^{N+1}),\ (x \to 0)$. Then, there exists a positive constant A, such that $|h(x)| \le A|x^{N+1}|,\ (x \to 0)$. Consider the limit:

$$\lim_{x\to 0} \frac{|h(x)|}{|x^N|} \le \lim_{x\to 0} \frac{A|x|^{N+1}}{|x|^N} = A \lim_{x\to 0} |x| = 0, \tag{2.52}$$

which proves the claim. Therefore, if $\varphi(x)$ is represented by a formal power series, we can write:

$$\varphi(x) = \sum_{n=0}^{N} a_n x^n + o(x^N),\ (x \to 0). \tag{2.53}$$

In particular, by repeating the same argument, but for the general asymptotic series in Def. 2.5, and by replacing $(x \to \infty)$ with $(x \to x_0)$, where x_0 is as in Def. 2.2[15], we arrive at a slightly different form of the asymptotic series:

Definition 2.6. *We say that the RHS of (2.54) is an asymptotic expansion to N terms of $\varphi(x)$:*

$$\varphi(x) = \sum_{n=0}^{N} a_n \zeta_n(x) + o(\zeta_N), \quad x \to x_0. \tag{2.54}$$

Both definitions 2.5 and 2.6 are useful. Which definition we use depends on the type of problem we have. We are now in a position to find a general formula for the coefficients a_n.

Theorem 2.4. *The coefficients $a_n, (n \ge 1)$ in (2.54) are given by:*

$$a_n = \lim_{x\to x_0} \frac{\left(\varphi(x) - \sum_{m=0}^{n-1} a_m \zeta_m(x)\right)}{\zeta_n(x)}. \tag{2.55}$$

[14]This can also be seen from (2.26).

[15]$(x \to \infty)$ was mainly chosen for convenience, but everything can easily be generalized for an arbitrary x_0.

Proof. The proof is very easy. Note that we changed the summed index from n to m simply for convenience. By definition of the o symbol, we have that:

$$0 = \lim_{x \to x_0} \frac{\Big(\varphi(x) - \sum_{m=0}^{n} a_m \zeta_m(x)\Big)}{\zeta_n(x)} = \lim_{x \to x_0} \frac{\Big(\varphi(x) - \sum_{n=0}^{n-1} a_m \zeta_m(x)\Big)}{\zeta_n(x)} - a_n. \tag{2.56}$$

Rearranging (2.56), we arrive at (2.55), as required. □

We can use (2.55) and (2.56) to prove uniqueness of the coefficients a_n as well. For this purpose, we need to first find a_0, expressed as a simple limit of two known function, namely φ and ζ_0, which we can deduce from (2.56) by plugging $n = 0$ into the first part. This proves the uniqueness of a_0 as the limit is unique. Then, using (2.55), we can find the next coefficient a_1, expressed as a limit, which includes a_0. Since, a_0 is unique, then so will a_1 be. Iterating this procedure, we see that the claim holds by obtaining the sequence:

$$\begin{cases} a_0 = \lim_{x \to x_0} \frac{\varphi(x)}{\zeta_0(x)} \\ a_1 = \lim_{x \to x_0} \frac{\varphi(x) - a_0 \zeta_0(x)}{\zeta_1(x)} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_n = \lim_{x \to x_0} \frac{\Big(\varphi(x) - \sum_{m=0}^{n-1} a_m \zeta_m(x)\Big)}{\zeta_n(x)}. \end{cases} \tag{2.57}$$

3 Watson's lemma

This section discusses integrals of the following type:

$$f(z) = \int_0^\infty e^{-zu} \varphi(u) du, \tag{3.1}$$

where the chosen function φ is such that the integral in (3.1) exists for a given $z \in \mathbb{C}$. Before dealing with more general functions, we first focus on functions of the form: $\varphi(u) = u^\lambda h(u)$, where $u, \lambda \in \mathbb{R},\ \lambda > -1$ (to ensure that the integral in (3.1) converges for $u = 0$) and $h(u)$ possesses a Taylor series in an open neighbourhood around $u = 0$ with $h(0) \neq 0$. If $h(u)$ has a zero of order m when $u = 0$, we can simply incorporate u^m into u^λ and define a new function h^*, such that $h^*(0) \neq 0$. If we also impose that $|h(u)| \leq De^{cu}$ over the integration range, where D, c are a positive constants, such that $|z| \geq c$, we have the following **Watson's lemma**:

Lemma 3.1.

$$\int_0^\infty e^{-zu} u^\lambda h(u) du = \lim_{n \to \infty} \Big(\sum_{m=0}^{n} \frac{h^m(0)\Gamma(m + \lambda + 1)}{m! z^{\lambda + m + 1}} + O(z^{-(\lambda + n + 2)}) \Big),\ |z| \to \infty,\ (|Arg(z)| < \frac{\pi}{2}). \tag{3.2}$$

Proof. The proof follows the discussion in [4], but we generalize it when the independent variable is complex and integration range is unbounded. There are several issues with the proof of this lemma. The main one is the complex variable z inside the integral. Another one is the fact that the Taylor series of $h(u)$ may not have an infinite radius of convergence. We deal with these problems one by one. First, let us split the integral in (3.2) in two parts (assuming that both integrals converge):

$$\int_0^\infty e^{-zu}u^\lambda h(u)du = \int_0^{U_1} e^{-zu}u^\lambda h(u)du + \int_{U_1}^\infty e^{-zu}u^\lambda h(u)du = I_{[0,U_1]} + I_{[U_1,\infty]}. \tag{3.3}$$

The notation $[0, U_1]$ means that we are integrating along the straight line from 0 to U_1 (in this case it happens to be the real line as U_1 is real). In (3.3) U_1 is a positive number, such that $U_1 < R$, where R is the radius of convergence of the Taylor series of h at $u = 0$. One might be wondering if we are dealing with convex subsets of $\mathbb{C}$. This is not true. h is a real function viewed over $\mathbb{R}$, but not over $\mathbb{C}$. While the integration range is a subset of $\mathbb{R}$, hence trivially convex, we are not integrating with respect to z, which can be taken from any subset of $\mathbb{C}$. However, we are going to assume that z belongs to a path-connected topological subspace (subset) of $\mathbb{C}$, so that the notion of the length of a curve and many others are properly defined. We will now focus on the first integral on the RHS of (3.3). Since $h(u)$ has a Taylor series at $u = 0$, we can write:

$$\begin{cases} h(u) = \sum_{m=0}^\infty \frac{h^m(0)}{m!}u^m = \sum_{m=0}^\infty a_m u^m = \sum_{m=0}^n a_m u^m + r_n(u), \\ |r_n(u)| < K u^{n+1}, \ for \ |u| < R, \end{cases} \tag{3.4}$$

where $r_n(u)$ is the remainder from the Taylor's theorem, K is a finite positive number and a_m are defined by equation (3.4). Now, since $U_1 < R$, we can plug the Taylor series expression of h into $I_{[0,U_1]}$ and split again the integral:

$$I_{[0,U_1]} = \int_0^{U_1} e^{-zu}u^\lambda \sum_{m=0}^n a_m u^m du + \int_0^{U_1} e^{-zu}u^\lambda r_n(u)du. \tag{3.5}$$

At this stage the reader might be thinking that we have not achieved much. The problem with the complex variable remains. However, when using the O symbol, we eventually took the moduli of the functions we considered. Let us try this approach here by first dealing with the second integral in (3.5):

$$\Big|\int_0^{U_1} e^{-zu}u^\lambda r_n(u)du\Big| \leq \int_0^{U_1} |e^{-zu}u^\lambda r_n(u)|du < K\int_0^{U_1} e^{-Re(z)u}u^{\lambda+n+1}du. \tag{3.6}$$

We now change the variables to $\xi = Re(z)u \Rightarrow d\xi = Re(z)du$. Hence,

$$\int_0^{U_1} e^{-Re(z)u}u^{\lambda+n+1}du = Re(z)^{-(\lambda+n+2)}\int_0^{Re(z)U_1} e^{-\xi}\xi^{\lambda+n+1}d\xi \tag{3.7}$$

$$= Re(z)^{-(\lambda+n+2)}\int_0^\infty e^{-\xi}\xi^{\lambda+n+1}d\xi - Re(z)^{-(\lambda+n+2)}\int_{Re(z)U_1}^\infty e^{-\xi}\xi^{\lambda+n+1}d\xi \tag{3.8}$$

$$= Re(z)^{-(\lambda+n+2)}\Gamma(\lambda+n+2) - Re(z)^{-(\lambda+n+2)} \int_{Re(z)U_1}^{\infty} e^{-\xi}\xi^{\lambda+n+1} d\xi. \tag{3.9}$$

One might argue that extending the integration range from $Re(z)U_1$ to ∞ is wrong. After all, we explicitly stated that we want to stay in an open neighbourhood, the diameter[16] of which, is less than (or equal to) R. However, at this stage I am working only with the polynomial functions, so extending the region is mathematically correct, as long as it reproduces the initial result, which it does, as we are using properties of the Riemann integral. We obtained the gamma function in the last inequality, which suggests that we are heading in the right direction. Let us try to bound the second integral in (3.9). If we decide to substitute the polynomial function with an exponential: $e^{\xi} > \xi,\ \forall \xi > 0$, we have:

$$\int_{Re(z)U_1}^{\infty} e^{-\xi}\xi^{\lambda+n+1} d\xi < \int_{Re(z)U_1}^{\infty} e^{-\xi} e^{(\lambda+n+1)\xi} d\xi = \int_{Re(z)U_1}^{\infty} e^{(\lambda+n)\xi} d\xi, \tag{3.10}$$

which is a divergent integral, and hence not particularly useful as we obtained that something finite is less than something infinite. However, by "adding" a few more polynomial functions, the jump to the exponential function is not that big and the procedure succeeds. Indeed, try $\xi = Re(z)U_1(1+v) \Rightarrow d\xi = Re(z)U_1 dv$:

$$Re(z)^{-(\lambda+n+2)} \int_{Re(z)U_1}^{\infty} e^{-\xi}\xi^{\lambda+n+1} d\xi = U_1^{\lambda+n+2} e^{-Re(z)U_1} \int_0^{\infty} e^{-Re(z)U_1 v}(1+v)^{\lambda+n+1} dv. \tag{3.11}$$

$$< U_1^{\lambda+n+2} e^{-Re(z)U_1} \int_0^{\infty} e^{-[Re(z)U_1-(\lambda+n+1)]v} dv, \tag{3.12}$$

where we used that $(1+v)^a < e^{av}$ for any $a, v > 0$. In order to get a convergent integral, we need to make sure that $Re(z)U_1 > (\lambda+n+1)$. At this stage comes the condition that $|z| \to \infty$. However, as the real part of z is appearing in (3.12), we also need to impose that $|Arg(z)| < \frac{\pi}{2}$. This implies that $Re(z) = |z|\cos(Arg(z)) > (\lambda+n+1)/U_1$ for sufficiently large $|z|$. Hence, the integral converges, which gives:

$$U_1^{\lambda+n+2} e^{-Re(z)U_1} \int_0^{\infty} e^{-[Re(z)U_1-(\lambda+n+1)]v} dv = U_1^{\lambda+n+2} e^{-Re(z)U_1} \frac{1}{Re(z)U_1 - (\lambda+n+1)} \tag{3.13}$$

$$= U_1^{\lambda+n+2} e^{-Re(z)U_1} \frac{1}{Re(z)U_1} \frac{1}{1 - \frac{\lambda+n+1}{Re(z)U_1}} = o(e^{-Re(z)U_1}),\ for\ |z| \to \infty\ ,\ |Arg(z)| < \frac{\pi}{2}. \tag{3.14}$$

The integral is exponentially decreasing. Putting everything together, we obtain:

$$\int_0^{U_1} e^{-Re(z)u} u^{\lambda+n+1} du = Re(z)^{-(\lambda+n+2)}\Gamma(\lambda+n+2) + o(e^{-Re(z)U_1}) \tag{3.15}$$

$$= O(Re(z)^{-(\lambda+n+2)}) = O(|z|^{-(\lambda+n+2)}), \quad when\ |z| \to \infty,\ |Arg(z)| < \frac{\pi}{2}. \tag{3.16}$$

[16] I am considering the induced topology $((0,\infty), \mathcal{T}_m)$ over $\mathbb{R}$, so the diameter will not be $2R$.

Now, since the LHS of (3.15) is the modulus of $\int_0^{U_1} e^{-zu}u^{\lambda+n+1}du$, we can generalize (3.15) and (3.16), using the definition of the O symbol, for $z \in \mathbb{C}$:

$$\int_0^{U_1} e^{-zu}u^{\lambda+n+1}du = O(z^{-(\lambda+n+2)}), \quad when\ |z| \to \infty\ and\ |Arg(z)| < \frac{\pi}{2} \Rightarrow \tag{3.17}$$

$$\int_0^{U_1} e^{-zu}u^{\lambda}r_n(u)du = O(z^{-(\lambda+n+2)}), \quad when\ |z| \to \infty\ and\ |Arg(z)| < \frac{\pi}{2}. \tag{3.18}$$

Let us now return back to the first integral in (3.5). By interchanging the sum and the integral (as the terms are finitely many) and by applying the same procedure as above, we easily get:

$$\int_0^{U_1} e^{-zu}u^{\lambda}\sum_{m=0}^{n} a_m u^m du = \sum_{m=0}^{n} a_m \Gamma(\lambda+m+1)z^{-(\lambda+m+1)} + O((z^{-(\lambda+n+2)}). \tag{3.19}$$

Therefore, substituting everything back to (3.5), we finally obtain:

$$I_{[0,U_1]} = \int_0^{U_1} e^{-zu}u^{\lambda}h(u)du = \sum_{m=0}^{n} \frac{h^m(0)}{m!}\Gamma(\lambda+m+1)z^{-(\lambda+m+1)} + O((z^{-(\lambda+n+2)})^{17}, \tag{3.20}$$

which upon iteration produces (3.2). This implies that $I_{[U_1,\infty]}$ needs to belong to the class $O((z^{-(\lambda+n+2)})$. The problem here is that we cannot substitute the Taylor series expression of h into the integral, since at some point we will go outside the open neighbourhood over which the series converges (i.e. when $u > R$). However, we know that $|h(u)| \leq De^{cu},\ \forall u > 0$, by assumption, and hence:

$$|I_{[U_1,\infty]}| = \Big| \int_{U_1}^{\infty} e^{-zu}u^{\lambda}h(u)du \Big| < \int_{U_1}^{\infty} e^{-Re(z)u}u^{\lambda}De^{cu}du. \tag{3.21}$$

We distinguish two cases now, namely $\lambda \in (-1, 0)$ and $\lambda > 0$ (the case when $\lambda = 0$ follows trivially). If $\lambda \in (-1, 0)$, then u^{λ} will be a decreasing function and since $Re(z) > c$, we estimate (3.21) by:

$$|I_{[U_1,\infty]}|_{\lambda\in(-1,0)} < D\int_{U_1}^{\infty} e^{-(Re(z)-c)u}U_1^{\lambda}du = -DU_1^{\lambda}e^{-(Re(z)-c)U_1} = O(e^{-Re(z)U_1}). \tag{3.22}$$

When $\lambda > 0$, then clearly $u^{\lambda} < e^{\lambda u}$ (an estimate that failed last time), gives:

$$|I_{[U_1,\infty]}|_{\lambda>0} < D\int_{U_1}^{\infty} e^{-(Re(z)-c-\lambda)u}du = O(e^{-Re(z)U_1}). \tag{3.23}$$

The key idea here (and before in the discussion, though being suppressed there) is that the class of functions $O(e^{-z})$ is contained in $O(z^{-N}),\ \forall N > 0$ when $|z| \to \infty$ (i.e. $O(e^{-Re(z)U_1}) = O(e^{-z}) = O((z^{-(\lambda+n+2)}))$. This implies that:

$$I_{[U_1,\infty]} = \int_{U_1}^{\infty} e^{-zu}u^{\lambda}h(u)du = O((z^{-(\lambda+n+2)}), \tag{3.24}$$

as required. □

[17] I should continue stating that this holds only when $|z| \to \infty$ with $|Arg(z)| < \frac{\pi}{2}$, but I hope that this is obvious.

4 Conclusion

The project achieved the purpose of introducing some simple concepts of the asymptotic methods. The theory developed will prove to be beneficial mainly for physicists looking for background material in order to start reading textbooks on this topic, such as [3]. A great emphasis is put on subtle connections between the subject and general topology, which is regarded useful in many other branches of physics, such as general relativity. What is more, some unique examples are provided to reinforce the introduced notions. A careful discussion about the differences between asymptotic series and power series from real or complex analysis is also present. Finally, sometimes several different proofs of the same theorem are provided and different points of view considered. Unique proofs take place on several occasions.

Building on the foundational material discussed here, one could explore important consequences of the Watson's lemma, for example the Laplace's method for integrals and saddlepoint approximations, which can immediately be applied in PDEs and many branches of physics. Nevertheless, the material covered focuses on issues and problems which many authors ignore, leaving a gap in the knowledge of the reader, who might otherwise struggle with the more important concepts that follow.

References

[1] John B. Conway. *Functions of One Complex Variable 1*. Springer, 1978.

[2] N. G. de Bruijn. *Asymptotic Methods in Analysis*. Dover Publications Inc., 2003.

[3] A. Erdelyi. *Asymptotic expansions*. Dover Publications Inc., 2003.

[4] J.D. Murray. *Asymptotic Analysis*. Springer, 1992.

[5] Frank Olver. *Asymptotics and Special Functions*. A K Peters/CRC Press, 1997.

[6] Carl M. Bender Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer, 1999.

[7] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Education Ltd., 1986.

[8] Wilson A. Sutherland. *Introduction to Metric and Topological spaces*. Oxford University Press, 2010.